Matrix Multiplications

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Matrix Multiplication

Dot product method

Vector $\rightarrow$ Column vector

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Dot product

$$x^T x = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= [1 + 4 + 9]$$

$$= 14$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix}$$ dot products

$$\underline{b} = x_1 c_1 + x_2 c_2 + \dots + x_n c_n$$

$$\underline{A}\,\underline{x} = \underline{b}$$

$A_{n \times n}$

$x \to$ vector

$b \to$ vector.

Matrix times a column is a column.

$$A = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$= 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

3] A row times a matrix A is a row.

It is linear combinations of rows of A.

$$\begin{bmatrix} 0 & 0 \end{bmatrix}_{1\times 2} \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} \; & \; \end{bmatrix}_{1\times 2} ?$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} + 2 \begin{bmatrix} 3 & 4 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 7 & 10 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$4$$

$$= \begin{bmatrix} 8 & 10 \\ 11 & 22 \end{bmatrix}.$$

$$2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 22 \end{bmatrix}$$

④. $X^T X$

$X X^T$

$$\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} + \cancel{\begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix}}$$

$$+ \cancel{\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix}} + \begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix} + \begin{bmatrix}2 & 4\\4 & 8\end{bmatrix} + \begin{bmatrix}3 & 4\\9 & 12\end{bmatrix} + \begin{bmatrix}6 & 8\\12 & 16\end{bmatrix}} \quad \begin{bmatrix}7 & 10\\15 & 22\end{bmatrix}$$

(5.) $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{20\times 20} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$

Application

1. System of Equations

$$Ax = b$$

$$A_{n\times n}\, x_{n\times 1} = b_{n\times 1}$$

$$Ax = b$$

$$\underline{A\,x} = \underline{\underline{b}}$$

$$Ax = [x_1 c_1 + x_2 c_2 \ldots + x_n c_n] \longrightarrow \text{Span}(c_1, c_2, \ldots c_n)$$

$$\underline{(1, 0, 1)}$$

$$A = [c_1 \; c_2 \ldots\ldots c_n]$$

$$\boxed{b \in cl\,(A)}$$

$$3x + 4y + z = 4$$
$$2x - 4y + 5z = \cancel{7}\,6$$
$$x + 3y + 3z = 4$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix} y + \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} z = \begin{bmatrix} 4 \\ \cancel{7}\,6 \\ 4 \end{bmatrix}$$

$$x + y = 2$$
$$3x + y = 4$$
$$5x + 3y = 8$$

$$\underline{(1, 1)}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\checkmark \ \boxed{b \in C(A)}$$

$$x + y + w = 2$$
$$3x + y + 2w = 4$$

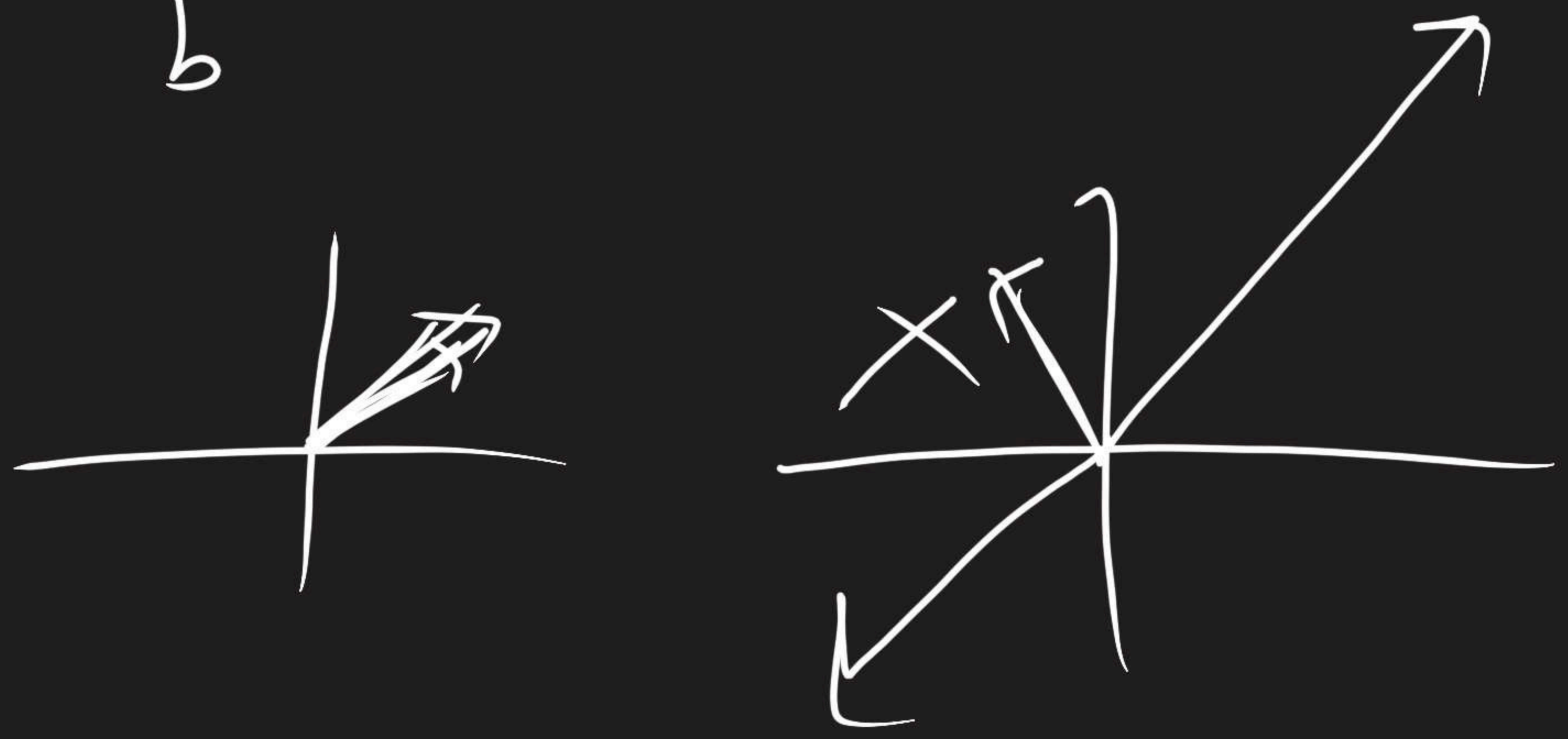
$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad (1, 1, 0)$$

2. Eigen values and Eigen vectors

$$Ax = \lambda x \quad , \quad \boxed{x \neq 0}$$

$\downarrow$

b



$$A_{m \times n} \, x_{n \times 1} = b_{m \times 1}$$

$$A(X) \rightarrow b_{m \times 1}$$

$\mathbb{R}^n \qquad \mathbb{R}^m$

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{matrix} c_1 & c_2 & c_3 \end{matrix}$$

$$\boxed{Ax = 0}$$

$$A\underline{x} = \lambda x$$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

Trace(A) = sum of λs $= 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Det(A) = products of e.values

$$3 = \boxed{0 + 0} + \lambda$$

3. Diagonalization

$\exists$ Invertible P s.t. $\boxed{P^{-1}AP = D}$

$P = [c_1 \;\; c_2 \;\; c_3]$ $\quad$ $\boxed{P^{-1}AP = D}$

$\boxed{\text{Independent}}$ Eigen vectors of A

$\lambda_1, \lambda_2, \lambda_3$

$AP = A[c_1 \;\; c_2 \;\; c_3]$

$Ax = x$ $\quad$ $\boxed{Ac_i = \lambda_i c_i}$

$= [Ac_1 \;\; Ac_2 \;\; Ac_3]$

$= [\lambda_1 c_1 \;\; \lambda_2 c_2 \;\; \lambda_3 c_3] = [c_1 \;\; c_2 \;\; c_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$