More on Groups...

MLS02: Group Theory
Theoretical Nexus

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Let's recall the Fundamental Theorem of Arithemetic, more intuitively known as the Unique Factorization theorem, which esentially states that any positive integer ≥ 1 can be written as a product of prime numbers in exactly one way (not referring to the order of the occurence of those numbers).

With this in mind, lets move ahead.

The Fundamental Theorem for Finite Abelian Groups

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Lets consider an example to understand this.

Consider an abelian group of order 60.
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Lets factorise 60, using the **Unique Factorization Theorem**...

$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$$

With this in mind...

The Fundamental Theorem for Finite Abelian Group states that if we wish to classify all abelian groups of order 60, we have the following possibilities:

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
- $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

Similarly, we can classify other finite abelian groups.

Fundamental Theorem of Finite Abelian Groups

Every finite abelian group G is isomorphic to a direct product of cyclic groups of the form

$$\mathbb{Z}_{p_1^{alpha_1}} \times \mathbb{Z}_{p_2^{alpha_2}} \times \cdots \times \mathbb{Z}_{p_n^{alpha_n}}$$

where the p_i's are primes (not necessarily distinct.)

p-groups

I will be borrowing the definition from Dummit and Foote here...

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A group of order p^{α} for some $\alpha \geq 1$ is called a *p*-group.

Subgroups of G which are *p*-groups are called *p*-subgroups.

Normal Series

A sequence of subgroups

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \{e\}$$

where each G_i is normal in G_{i-1}.

Subnormal Series

A sequence of subgroups $G = G_0 > G_1 > G_2 > \cdots > G_n = \{e\}$
where each is a subnormal subgroup of .

Solvable Groups

A group G is considered to be solvable if it possesses a normal series where each factor group (i.e. quotient group) is abelian.

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So, I think we are comfortable with abelian groups and what we mean by those is essentially that every elemnt of an abelian group is commutative with every other element of the group.

This implies that each factor group of the abelian group will also be abelian. And hence, all abelian groups are solvable.

Consider the group D_8 (following the D_{2n} notation), the dihedral group of order 8, which represents the symmetries of a square. These symmetries include:

The group D_4 is solvable, if we can break it down into simpler, abelian groups.

Can we do so?

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Every finite group of order < 60, every Abelian group, and every subgroup of a solvable group is <u>solvable</u>.

Soluble Groups

It's just another name for solvable groups!

Sylow p-Groups

If G is a group of order $p^{\alpha}m$, where p does not divide m, then a subgroup of order p^{α} is called a Sylow p-subgroup of G, or a p-Sylow subgroup of G.

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An example perhaps?

Lets consider a group of order 45.

We can make this more clear, I think. Let us consider a group of order **450**?

"You have to ignore low-hanging fruit, which is a little tricky. I'm not sure if it's the best way of doing things, actually—you're torturing yourself along the way. But life isn't supposed to be easy either."

-Mirzakhani

Thank you.