The Basics of Set Theory

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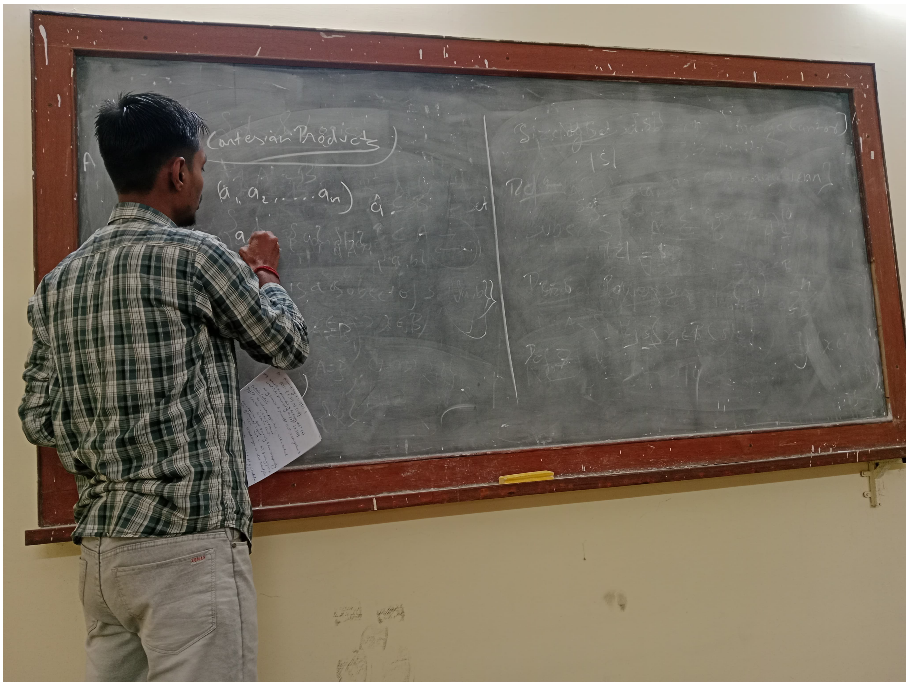


Krishnakant while Lecture

1 Basics of Sets

The language of sets is a means to study such collections in an organized fashion.
Let's see the definition: **[Intuitive]** [Not a part of formal theory of sets].

Definition 1: A **set** is an unordered collection of distinct objects called **elements or members of the set**.
To take an element a in set A, we write:

$$a \in A, \quad a \notin A \quad \text{[denotes not belonging]}$$

Lowercase letters are often used for elements, while Uppercase are generally denoted as **Names of the Sets**.

Several Ways to Represent Sets:

1. **Roster Form:** One way is to list all the members of the set,when this is possible. we use a notation where all members os the set are listed between braces.

The set V of all vowels in the English alphabet can be written as V= $\{a, e, i, o, u\}$

Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then ellipses $(\ldots)$ are used when the general pattern of the elements is obvious.

No less than 100: $A = \{1, 2, 3, \ldots, 99\}$.

2. Another way is to use **Set Builder Notation:**

General form: $\{x \mid x \text{ has property } P\}$.

For instance:

$$O = \{x \mid x \text{ is an odd integer less than } 10\}.$$

Some Common Sets You Must Know:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of all natural numbers,

- $\mathbb{Z} = \{\cdots, 2, 1, 0, 1, 2, \cdots\}$, the set of all integers
  $Z^+ = \{1, 2, 3, \ldots\}$, the set of all positive integers

- $\mathbb{Q} = \left\{\frac{p}{q} \mid p \in \mathbb{Z},\, q \in \mathbb{Z},\, q \neq 0\right\}$, the set of all rational numbers.

- $\mathbb{R}$, the set of all Real numbers,

- $\mathbb{R}^+$, the set of all Positive Real numbers,

- $\mathbb{C}$, the set of all Complex numbers,

Intervals in Calculus: Sometimes, we use subsets like intervals to represent the set of two real numbers a and b between them:

$$[a, b] \text{ (Closed Interval)}, \quad (a, b) \text{ (Open Interval)}.$$

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, b) = \{x \mid a < x < b\}$$

Sets can have other sets as their elements or members.

For Example: The set N, Z, Q, R is a set containing four elements, each of which is a set. The four elements of this set are N, the set of natural numbers; Z, the set of integers; Q, the set of rational numbers; and R, the set of real numbers.

Definition 2: Two sets are **equal** if and only if they have the same elements.
Therefore, if A and B are sets, then A and B are equal if and only if

Mathematical Representation:

$$A = B \iff \forall x(x \in A \leftrightarrow x \in B)$$

We write A = B if A and B are equal sets.

Null/Empty Set ($\emptyset$):

The set having no element, denoted by $\emptyset$ or $\{\}$.

Singleton Set:

The set with only one element: $\{B\}$.
A single element of the set is the empty set itself.
Example: A useful analogy for remembering this difference is to think of folders in a computer file system.
The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder

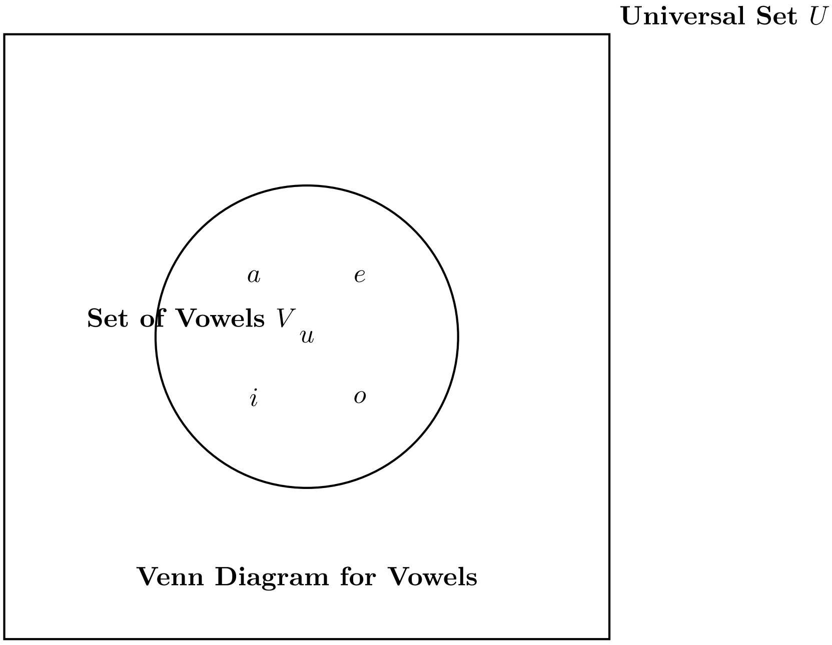
2 Naive Set Theory:

George Cantor's Original Set Theory.
We will be going with this theory.
May get familiarity with *Axiomatic Set Theory* if we go further.

Venn Diagram:

Graphical Representation of Sets.
Universal Set U
Often used to indicate the relation between sets.

Example: A diagram showing $A \subseteq U$ inside a circle.



We may encounter situations where the **elements of one set are also elements of a second set.** Let us introduce some terminology and notation to express such relationships.

Defintion 3: Subsets:

A set is a **subset** of B if and only if all elements of A are in set B.
A is in set B:

$$A \subseteq B \quad \text{[A is the subset of B]}$$

$$\forall x(x \in A \implies x \in B)$$

Showing A is **not** a subset of B:

$$A \nsubseteq B$$

A subset itself is a subset of B.

Proper Subset:

We wish to emphasize: Set A is a subset of B but $A \neq B$:

$$A \subset B$$

$$\forall x(x \in A \implies x \in B) \land \exists x(x \in B \land x \notin A)$$

Showing two sets are **equal**. If $A = B$, show that:

$$A \subseteq B,\ B \subseteq A$$

For instance:

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \quad B = \{x \mid x \text{ is a subset of } \{a, b\}\}$$

$$A = B$$

Note: $\{a\} \in A$ *but* $a \notin A$.

3 The Size of a Set:

Definition 4: Finite set :

Let S be a set, and it has exactly n elements in S.
n is a non-negative integer, we say that S is a **finite set**, and n is the **cardinality** (denoted by $|S|$).
Example: S = Set of odd true integers less than 10:

$$|S| = 5$$

Definition 5: Infinite Set

A set is said to be **infinite** if it is not finite.
Example: Set of all positive integers.

Definition 6: Power Sets:

S is a **power set** of T—the set of all subsets of T.

$$\boxed{\text{Power Set of } S\text{: Denoted by } P(S).}$$

Example: Power Set of $\{1, 2\}$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

Question:

What is the power set of the empty set?

$$P(\emptyset) = \{\emptyset\}$$

What is the power set of $\{\emptyset\}$?

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

The number of elements in the power set is 2^n, where n is the number of elements.

4 Cartesian Products:

Definition 7: The ordered n-tuple:

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is an **ordered collection**, where a_1 is the first element, and a_n is the nth element.
Two n-tuples are **equal** if and only if their corresponding pairs of elements are equal.

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \iff a_i = b_i \,\forall i.$$

In particular, ordered 2-tuples are called **ordered pairs**.

4.1 Definition 8:Cartesian Products:

Let A and B be sets. The Cartesian product $[A \times B]$ is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Example:
Let $A = \{1, 2\}, B = \{a, b, c\}$.

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

4.2 Definition 9:Cartesian product of n sets

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Question: What is the Cartesian product of $A = \{0, 1\}, B = \{S, T\}, C = \{\emptyset, 2\}$?

A^2 denotes the Cartesian product with itself.

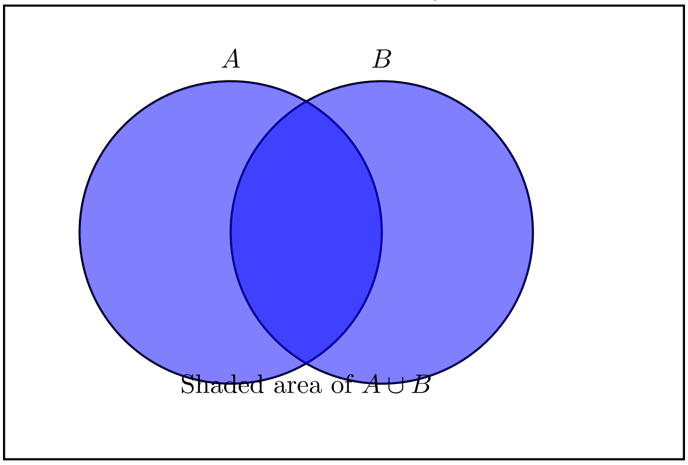
5 Set Operations

Definition 1:

Let A and B be two sets. The **union** of A and B, denoted $A \cup B$, is the set containing those elements either in A or in B.

Venn Diagram:

Universal Set U

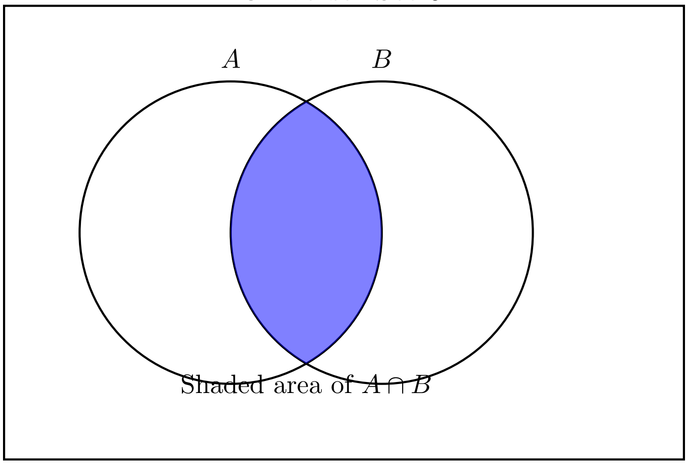


Definition 2:

$A \cap B$ (**Intersection**) is the set containing those elements in both A and B.

Venn Diagram:

Universal Set U



Definition 3:

Disjoint Sets – If $A \cap B$ is an empty set.

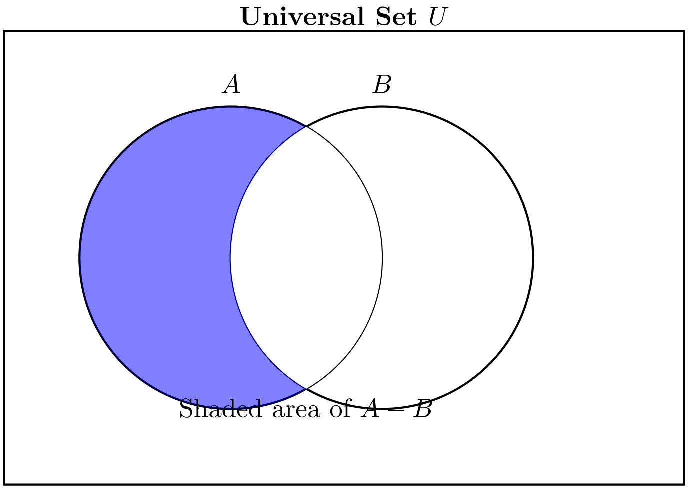
Definition 4:

The **difference** $A - B$ is the set of elements that are in A but not in B.

$$A - B = \{x \mid x \in A \land x \notin B\}$$

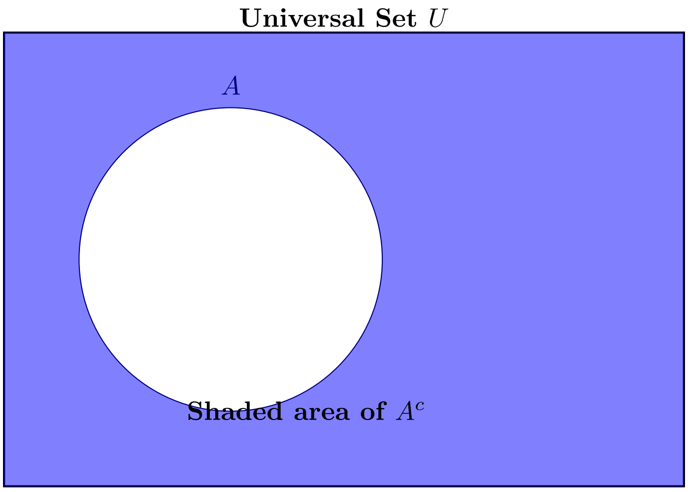
Sometimes also denoted as $A \setminus B$.

Venn Diagram



Definition 5:

The **complement** of a set A is denoted by A^c or $\overline{A}$.



6 Set Identities:

TABLE 1 Set Identities.	
Identity	**Name**
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws