Lie Groups, Class Groups and Category Theory

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Lie Groups

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A smooth (means C^∞) differentiable manifold M (real or complex). When G is a group too. Then it is called Lie group. Group of continuous symmetries.

Topological Groups: Groups $+$ Topological Spaces (all the operations are continuous).

Group structure: G has a group structure with a binary operation μ. So $\mu : G \times G \to G$.

Manifold structure: G is a topological space and admits differention.

Smooth binary operations yield Lie groups. Examples like SO(2) rotation in 2 dimenions, SU(n), U(1), Lorentz groups, Poincare groups, E_8, E_6 etc.

One can also have p-adic Lie groups, with metric completion of $\mathbb{Q}$.

PID and Class Group

In a ring R, when every ideal is principal, then it is called Principal Ideal Domain.

Theorem

Every PID is unique factorization domain.

Proof.

Easy to proof for $\mathbb{Z}$. Refer to any text in commutative algebra. $\square$

Whena a ring is not PID, then UFD is not guaranteed.

Class group is a measure of how the ring of integers in a field K fails to be unique factorization domain. It is J_K/P_K quotient group. J_K are group fractional ideals and P_K subgroup of principal ideals.

The order of this (ideal) class group is called class number. If class number 1, then UFD.

Category Theory

A category $\mathcal{C}$ consists of

1. Objects
2. Morphisms

A category of groups *Grp* consists of groups as objects and morphisms between them become group homomorphisms. There exists kernel (in category theoretical sense).

Initial object and final object of category. If both, then called zero object of category. Zero objects of *Grp* are groups with only identity.

A functor between two categories. From *Grp* to *Mon* (group of monoids).