Inner Product and Dirac's Bra Ket Notation

Devang Bajpai

Department of Physics,
CSJM University, Kanpur,
Uttar Pradesh 208012, India

Abstract

These notes aim to provide an introduction to the basics of inner products and Dirac's Bra Ket notation. These notes are divided into two parts in first part we will talk about inner product on $\mathbb{R}^n$ and $\mathbb{C}^n$, orthonormal bases, Gram-Schmidt procedure, orthogonal projectors. In second part we will talk about Dirac bra-ket notation for inner products.

Contents

1 Inner Product **1**

1.1 Inner Product in $\mathbb{R}^n$ 1

1.2 Inner Product in $\mathbb{C}^n$ 2

2 Orthonormal Basis and Orthogonal Projectors **2**

3 From inner products to bra-kets **4**

1 Inner Product

1.1 Inner Product in $\mathbb{R}^n$

An **inner product** on a vector space V over $\mathbb{F}$ is a machine that takes an ordered pair of elements of V , that is, two vectors, and yields a number in $\mathbb{F}$. In order to motivate the definition of an inner product we first discuss the case of real vector spaces and begin by recalling the way in which we associate a length to a vector.

The length of a vector, or **norm** of a vector, is a real non-negative number, equal to zero if the vector is the zero vector. In $\mathbb{R}^n$ a vector $a = (a_1, \ldots, a_n)$ has norm $\|a\|$ defined by

$$\|a\| = \sqrt{a_1^2 + \ldots + a_n^2} \tag{1.1}$$

Squaring this we view $\|a\|^2$ as the dot product of a with a:

$$\|a\|^2 = a \cdot a = a_1^2 + \ldots + a_n^2 \tag{1.2}$$

In order to generalize this dot product we require the following properties:

1. $a \cdot a > 0, \quad \forall \quad a$
2. $a \cdot a = 0 \quad \Longleftrightarrow \quad a = 0$
3. $a \cdot (b_1 + b_2) = a \cdot b_1 + a \cdot b_2$.
4. $a \cdot (\alpha b) = \alpha a \cdot b$, with $\alpha \in \mathbb{R}$ a number.
5. $a \cdot b = b \cdot a$.

The above axioms guarantee a fundamental result called the Schwarz inequality:

$$|a \cdot b| \leq |a||b| \tag{1.3}$$

Note that on the left-hand side the bars denote absolute value while on the right-hand side they denote norm.

1.2 Inner Product in $\mathbb{C}^n$

An inner product on a vector space V over $\mathbb{F}$ is a map from an ordered pair (u, v) of vectors in V to a number $\langle u, v\rangle$ in $\mathbb{F}$. The axioms for $\langle u, v\rangle$ are inspired by the axioms we listed for the dot product.

1. $\langle v, v\rangle \leq 0, \quad \forall \quad v \in V$
2. $\langle v, v\rangle = 0 \quad \Longleftrightarrow \quad v = 0$
3. $\langle u, v_1 + v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle$.
4. $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle,$ with $\alpha \in \mathbb{F}$
5. $\langle u, v\rangle = \langle v, u\rangle^*$

The norm $\|v\|$ of a vector $v \in V$ is defined by relation

$$\|v\|^2 = \langle v, v\rangle \tag{1.4}$$

Two vectors $u, v \in V$ are said to be **orthogonal** if $\langle u, v\rangle = 0$. This, of course, means that $\langle v, u\rangle = 0$ as well. The zero vector is orthogonal to all vectors (including itself). The inner product we have defined is **non-degenerate**: any vector orthogonal to all vectors in the vector space must be equal to zero. Indeed, if $x \in V$ is such that $\langle x, v\rangle = 0$ for all v, pick $v = x$, so that $\langle x, x\rangle = 0$ implies $x = 0$ by axiom 2.

2 Orthonormal Basis and Orthogonal Projectors

A list of vectors is said to be **orthonormal** if all vectors have norm one and are pairwise orthogonal. If $(e_1, \ldots, e_n)$ is a list of orthonormal vectors in V then

$$\langle e_i, e_j\rangle = \delta_{ij} \tag{2.5}$$

We also have a simple expression for the norm of $a_1e_1 + \ldots + a_ne_n$, with $a_i \in \mathbb{F}$:

$$\|a_1e_1 + \ldots + a_ne_n\|^2 = \langle a_1e_1 + \ldots + a_ne_n, a_1e_1 + \ldots + a_ne_n\rangle \tag{2.6}$$

$$= \langle a_1e_1, a_1e_1\rangle + \ldots + \langle a_ne_n, a_ne_n\rangle \tag{2.7}$$

$$= \|a_1\|^2 + \ldots + \|a_n\|^2 \tag{2.8}$$

This result implies the somewhat nontrivial fact that the vectors in any orthonormal list are linearly independent. Indeed if $a_1e_1 + ... + a_ne_n = 0$ then its norm is zero and so is $\|a_1\|^2 + \ldots + \|a_n\|^2$. This implies all $a_i = 0$, thus proving the claim. An **orthonormal basis** of V is a list of orthonormal vectors that is also a basis for V. Let $(e_1, \ldots, e_n)$ denote an orthonormal basis. Then any vector v can be written as

$$v = a_1e_1 + \ldots + a_ne_n, \tag{2.9}$$

for some constants ai that can be calculated as follows

$$a_i = \langle e_i, v\rangle \tag{2.10}$$

Indeed,

$$\langle e_i, v\rangle = \sum_j \langle e_i, a_je_j\rangle = \sum_j a_j\langle e_i, e_j\rangle = \sum_j a_j\delta ij = a_i. \tag{2.11}$$

Therefore any vector v can be written as

$$v = \langle e1, v\rangle e_1 + \ldots + \langle e_n, v\rangle e_n = \sum_i \langle e_i, v\rangle e_i \tag{2.12}$$

. To find an orthonormal basis on an inner product space V we can start with any basis and then follow a certain procedure. A little more generally, we have the **Gram-Schmidt** procedure: Given a list $(v_1, \ldots, v_n)$ of linearly independent vectors in V one can construct a list $(e_1, \ldots, e_n)$ of orthonormal vectors such that both lists span the same subspace of V. An inner product can help us construct interesting subspaces of a vector space V . Consider any subset U of vectors in V. Then we can define a subspace $U^\perp$, called the **orthogonal complement** of U as the set of all vectors orthogonal to the vectors in U:

$$U^\perp = \{v \in V \mid \langle v, u\rangle = 0, \text{ for all } u \in U\}. \tag{2.13}$$

This is clearly a subspace of V . When the set U is itself a subspace, then U and $U^\perp$ actually give a direct sum decomposition of the full space:

Theorem 1 *If U is a subspace of V, then $V = U \oplus U^\perp$.*

One can define a linear operator P_U , called the **orthogonal projection** of V onto U, that acting on v above gives the vector $u : P_Uv = u$. It is clear from this definition that:

1. range$P_U = U$. Thus P_U is not surjective.

2. null$P_U = U^{\perp}$. Thus P_U is not invertible.

3. P_U acting on U is the identity operator. Thus if we act twice with P_U on a vector, the second action has no effect as it is acting on a vector in U. Thus

$$P_U P_U = P_U^2 = P_U \tag{2.14}$$

4. $|P_U v| \leq |v|$. The action of P_U cannot increase the length of a vector. This follows from the decomposition and the Pythagorean theorem:

$$|v|^2 = |u+w|^2 = |u|^2 + |w|^2 \geq |u|^2 = |P_U v|^2, \tag{2.15}$$

and taking the square root.

5. $\det P_U = 0$

6. tr$P_U = n = \text{dim}n$.

3 From inner products to bra-kets

It all begins by writing the inner product differently. The first step in the Dirac notation is to turn inner products into so called "bra-ket" pairs as follows

$$\langle v, u \rangle \to \langle v | u \rangle$$

Instead of the inner product comma we simply put a vertical bar! The object to the right of the arrow is called a bra-ket. Here the symbol $|v\rangle$ is called a ket and the symbol $\langle v|$ is called a bra. The bra-ket is recovered when the space between the bra and the ket collapses. Since things look a bit different in this notation let us re-write a few of the properties of inner products in bra-ket notation.

1. $\langle v|v\rangle \leq 0, \quad \forall \quad v \in V$

2. $\langle v|v\rangle = 0 \quad \iff \quad v = 0$

3. $\langle u|v_1 + v_2\rangle = \langle u|v_1\rangle + \langle u|v_2\rangle$.

4. $\langle u|\alpha v\rangle = \alpha\langle u|v\rangle$, with $\alpha \in \mathbb{F}$

5. $\langle u|v\rangle = \langle v|u\rangle^*$

Sometimes the label inside a ket is the vector itself, other times it is a quantity that characterizes the vector. Bras are rather different from kets although we also label them by vectors. Bras are linear functionals on the vector space V . The set of all linear functionals on V is in fact a new vector space over $\mathbb{F}$, the vector space V^* dual to V.

Note here that if the label in the ket is not a vector; it is the position on a line, or a state in an infinite dimensional complex vector space. Therefore, the following should be noted

$$|ax\rangle \neq a\,|x\rangle\,, \text{ for any real } a = 1,$$

$$|{-x}\rangle \neq -|x>, \text{ unless } x = 0,$$

$$|x_1 + x_2\rangle \neq |x_1\rangle + |x_2\rangle\,,$$

All these equations would hold if the labels inside the kets were vectors.