Linear Vector Spaces

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MLS01

Madhava Lecture Series, Theoretical Nexus

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Contents

1 Motivation to Vector Spaces

Review 12th Class

Let the vectors be:

$$\vec{a} = a_1\hat{i} + a_2\hat{j}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j}$$

Vector Addition

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j}$$

Scalar Multiplication

If $\alpha \in \mathbb{R}$, then:

$$\alpha\vec{a} = \alpha a_1\hat{i} + \alpha a_2\hat{j}$$

For example, let $\vec{a} = 2\hat{i} + 4\hat{j}$ and $\alpha = 3$. Then:

$$\alpha\vec{a} = 6\hat{i} + 12\hat{j}$$

1.1 Properties of Vector Spaces

1. **Commutativity:** $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

2. **Associativity of Addition:** $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

3. **Existence of Zero Vector:** There exists a zero vector $\vec{0} = 0\hat{i} + 0\hat{j}$ such that:

$$\vec{a} + \vec{0} = \vec{a} \quad \text{and} \quad \vec{b} = \vec{0}$$

4. **Existence of Additive Inverse:** For all vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, there exists $-\vec{a} = (-a_1)\hat{i} + (-a_2)\hat{j} + (-a_3)\hat{k}$ such that:

$$\vec{a} + (-\vec{a}) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

5. **Distributivity of Scalar Multiplication:** $\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}$

6. **Distributivity over Scalars:** $(\alpha + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$

7. **Associativity of Scalars:** $(\alpha\beta)\vec{a} = \alpha(\beta\vec{a})$

8. **Multiplicative Identity:** $1 \cdot \vec{a} = 1a_1\hat{i} + 1a_2\hat{j} = \vec{a}$

2 What is Field?

A **non-empty set** F **together with two binary operations** $+$ (*called addition*), $\cdot$ (*called multiplication*) is said to be a **field** if the following conditions are satisfied:

1. $a + b = b + a \quad \forall\, a, b \in F$ {Commutativity of Addition}
2. $a \cdot b = b \cdot a \quad \forall\, a, b \in F$ {Commutativity of Multiplication}

3. $\forall\, a, b, c \in F$:

$$(a + b) + c = a + (b + c)\{Associativity of Addition\}$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)\{Associativity of Multiplication\}$$

4. **Existence of addition and multiplicative identity:**
   - $\exists\, 0 \in F$ such that $a + 0 = a \quad \forall\, a \in F$
   - $\exists\, 1 \in F$ such that $a \cdot 1 = a \quad \forall\, a \in F$
5. **Existence of additive and multiplicative inverses:**
   - $\forall\, a \in F,\ \exists\, b \in F$ such that $a + b = 0$ *(Note: b is denoted by $-a$)*
   - $\forall\, a \in F, a \neq 0,\ \exists\, b \in F$ such that $a \cdot b = 1$ *(Note: b is denoted by a^{-1})*
6. **Distributivity:**

$$a \cdot (b + c) = ab + ac$$
$$(a + b) \cdot c = ac + bc \quad \forall\, a, b, c \in F$$

Note: Closure property holds for all operations.

Example of Finite Field

Ex: Let p be **prime**.

$$\mathbb{Z}_p = \{0, 1, 2, 3, \ldots, p - 1\}$$

and binary operation:

1. $a + b =$ least non-negative remainder when $a + b$ is divided by p.

$$(a + b) = (a + b) \bmod p$$

2. $ab =$ least non-negative remainder when ab is divided by p.

$$ab = (ab) \bmod p$$

$(\mathbb{Z}_p, +, \cdot)$ is a field.

Note: p is prime.

Fields

- $(\mathbb{R}, +, \cdot)$
- $(\mathbb{Q}, +, \cdot)$
- $(\mathbb{Z}_p, +, \cdot)$ where p is prime
- $(\mathbb{Z}_3, +, \cdot)$ where $\mathbb{Z}_3 = \{0, 1, 2\}$

Example: $(\mathbb{Z}_3, +, \cdot)$ where $p = 3$.

- Addition table:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

- Multiplication table:

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Note: In $\mathbb{Z}_p$,

$$a + b = (a + b) \bmod p, \quad ab = (ab) \bmod p.$$

3 Vector Spaces

Let F be any field, and V be a non-empty set. We say that $V(\mathbb{F})$ is a **vector space over** F if we can define operations:

1. **Vector Addition:** Denoted by $+ : \forall u, v \in V,\ u + v \in V$ (unique element).
2. **Scalar Multiplication:** $\forall \alpha \in F,\ \forall v \in V,\ \alpha \cdot v \in V$ (unique element).

Such that the following conditions are satisfied:

(A1) $u + v = v + u \quad \forall u, v \in \mathbb{V}$.

(A2) $(u + v) + w = u + (v + w) \quad \forall u, v, w \in \mathbb{V}$.

(A3) $\exists\, 0 \in V$ such that $u + 0 = u \quad \forall u \in \mathbb{V}$.

(A4) $\forall u \in \mathbb{V},\ \exists\, y \in \mathbb{V}$ such that $u + y = 0$.
(Note: y is denoted by $-u$).

(S1) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \quad \forall \alpha \in \mathbb{F},\ u, v \in \mathbb{V}$.

(S2) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v \quad \forall \alpha, \beta \in \mathbb{F},\ v \in \mathbb{V}$.

(S3) $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v) \quad \forall \alpha, \beta \in \mathbb{F},\ v \in \mathbb{V}$.

(S4) $1 \cdot v = v \quad \forall v \in \mathbb{V}$.

Important Notes

- $V(\mathbb{F})$is a **vector space over** $\mathbb{F}$. We write $V(\mathbb{F})$ as **a vector space**.
- If $V(\mathbb{F})$ is a vector space, then elements of V are called **vectors**, and elements of$\mathbb{F}$ are called **scalars**.

4 Properties of Vector Space

If $V(\mathbb{F})$ is a vector space:

1. $0 \cdot v = 0 \quad \forall v \in \mathbb{V}$.
2. $\alpha \cdot 0 = 0 \quad \forall \alpha \in \mathbb{F}$.
3. $(-\alpha) \cdot v = -(\alpha \cdot v) \quad \forall \alpha \in \mathbb{F},\ v \in \mathbb{V}$.
4. $\alpha \cdot (-v) = -(\alpha \cdot v) \quad \forall \alpha \in \mathbb{F},\ v \in \mathbb{V}$.
5. $(-\alpha) \cdot (-v) = \alpha \cdot v \quad \forall \alpha \in \mathbb{F},\ v \in \mathbb{V}$.

Key Points

- $V(\mathbb{F})$ is a vector space if:
  - $\mathbb{V}$ is a non-empty set.
  - $\mathbb{F}$ is a field.
  - There exists unique operations (vector addition and scalar multiplication) sat-isfying all axioms.
- Every field $\mathbb{F}$ is a vector space over itself.

5 Subspace of a Vector Space

Let $V(\mathbb{F})$ be any vector space, and W be a non-empty subset of V. Then W is said to be a **subspace** of V if:

1. $0 \in W$ (*Zero element of V is in W*).
2. $u, v \in W \implies u + v \in W$.
3. $u \in W \implies -u \in W$.
4. $\alpha \in F, u \in W \implies \alpha u \in W$.

Example

Let $V = \mathbb{R}^3$ (i.e., $\{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$) and $F = \mathbb{R}$. Define:

$$W = \{(a, 0, a) : a \in \mathbb{R}\}$$

- **Vector Addition:** $(a_1, 0, a_1) + (a_2, 0, a_2) = (a_1 + a_2, 0, a_1 + a_2)$.
- **Scalar Multiplication:** $\alpha(a, 0, a) = (\alpha a, 0, \alpha a)$.

Since $V(\mathbb{F})$ is a vector space, from (i), (ii), (iii), and (iv), W is a **subspace of** $\mathbb{V}$.

Vector spaces are subspaces of fields (some field $\mathbb{F}$).

Result

If $V(\mathbb{F})$ is a vector space and $\phi \neq W \subseteq V$, then W is a subspace of $\mathbb{V}$ if and only if:

1. W is a vector space w.r.t. vector addition and scalar multiplication that make$V(\mathbb{F})$ a vector space.

5.1 Subspace Test-1

Let$V(\mathbb{F})$ be a vector space, and $\phi \neq W \subseteq V$. Then W is a subspace of V if and only if:

1. $x, y \in W \implies x + y \in W$.
2. $\alpha x \in W \quad \forall \alpha \in F, x \in W$.

5.2 Sub-field of a Field

Let $\mathbb{F}$ be any field, and K a non-empty subset of $\mathbb{F}$. $\mathbb{F}$ is said to be a **sub-field** of $\mathbb{F}$ if:

- $\forall\ a, b \in K \implies a - b \in K$.
- $\forall\ a, b \in K, b \neq 0 \implies a.b^{-1} \in K$.

Example

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

1. Let $x = a_1 + b_1\sqrt{2}$ and $y = a_2 + b_2\sqrt{2}$.

$$x - y = (a_1 - a_2) + (b_1 - b_2)\sqrt{2} \in \mathbb{Q}(\sqrt{2}) \quad \text{(since } a_1 - a_2 \in \mathbb{Q} \text{ and } b_1 - b_2 \in \mathbb{Q}\text{)}.$$

2. If $y \neq 0 \implies a_2 \neq 0$ or $b_2 \neq 0$, then:

$$\frac{x}{y} = \frac{a_1 + b_1\sqrt{2}}{a_2 + b_2\sqrt{2}} \times \frac{a_2 - b_2\sqrt{2}}{a_2 - b_2\sqrt{2}}$$

Simplifying:

$$\frac{x}{y} = \frac{(a_1a_2 - b_1b_2) + \sqrt{2}(a_2b_1 - b_2a_1)}{a_2^2 - 2b_2^2} \in \mathbb{Q}(\sqrt{2}).$$

- $\mathbb{Q}\sqrt{2}$ is sub-field of $\mathbb{Q}\sqrt{2}$.
- $\mathbb{Q}\sqrt{2}$ is sub-field of $\mathbb{Q}(i)$.
- $\mathbb{Q}$ is sub-field of $\mathbb{R}$.
- $\mathbb{Q}\sqrt{2}$ is sub-field of $\mathbb{R}$.
- $\mathbb{R}, \mathbb{Q}, \mathbb{Q}\sqrt{2}$ is sub-field of $\mathbb{C}$.
- $\mathbb{Q}(i)$ is sub-field of $\mathbb{C}$.

6 Space of n-Tuples

Let $\mathbb{F}$ be any field and k be a sub-field of $\mathbb{F}$. Define:

$$V = F^n = \{(a_1, a_2, a_3, \dots, a_n) \mid a_1, a_2, \dots, a_n \in F\}.$$

Then $V(F)$ is a vector space if:

1. **Vector Addition:**
$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$
2. **Scalar Multiplication:**
$$\alpha \in k, \quad \alpha(a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n).$$
3. **Zero Vector:**
$$O = (0, 0, 0, \dots, 0).$$

Examples:

- $F = \mathbb{R}$, $k = \mathbb{R}$
  - $\mathbb{R}^n(\mathbb{R})$ is a vector space.
    * $\mathbb{R}^1(\mathbb{R})$ is a vector space.
    * $\mathbb{R}^2(\mathbb{R})$ is a vector space.
    * $\mathbb{R}^3(\mathbb{R})$ is a vector space. , $\quad \dots$
  - $\mathbb{R}^n = \{(a_1, a_2, a_3, \dots, a_n) \mid a_1, a_2, a_3, \dots, a_n \in \mathbb{R}\}$.
    * $\mathbb{R}^2 = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$.
    * $\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}$.
    * $\mathbb{R}^4 = \{(a_1, a_2, a_3, a_4) \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$.
- $F = \mathbb{C}$, $k = \mathbb{C}$:
  $\mathbb{C}^n(\mathbb{C})$ is a vector space.
  $\mathbb{C}(\mathbb{C})$ is a vector space.
  $\mathbb{C}^2(\mathbb{C})$ is a vector space.
- $F = \mathbb{C}$, $k = \mathbb{R}$:
- $F = \mathbb{C}$, $k = \mathbb{Q}$:

7 Space of Matrices

Let $\mathbb{F}$ be any field and k be a sub-field of $\mathbb{F}$. Define:

$$V = F^{m\times n} = \{A = (a_{ij})_{m\times n} \mid a_{ij} \in F \quad \forall\, i = 1, 2, \dots, m,\ j = 1, 2, \dots, n\}.$$

Then $\mathbb{V}$is a vector space over k, w.r.t.:

1. **Vector Addition:**
$$(a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}.$$
2. **Scalar Multiplication:**
$$\alpha(a_{ij})_{m\times n} = (\alpha a_{ij})_{m\times n}.$$
3. **Zero Matrix:**
$$O = (0)_{m\times n}.$$

Note

- $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$
- $\mathbb{C}^2 = \{(a_1, b_1), (a_2, b_2) \mid a_1, b_1, a_2, b_2 \in \mathbb{F}\}$
- $\mathbb{C}^2 = \{(a_1 + ib_1, a_2 + ib_2) \mid a_1, a_2 \in \mathbb{R},\ b_1, b_2 \in \mathbb{R}\}$
- $\mathbb{C}^3 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3) \mid a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{F}\}$
- $\mathbb{C}^3 = \{(a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) \mid a_1, a_2, a_3 \in \mathbb{R},\ b_1, b_2, b_3 \in \mathbb{R}\}$
- $\mathbb{C}^2(\mathbb{Q}[\sqrt{2}])$ or $(a + b\sqrt{2}, a_1 + b_1\sqrt{2})$ $(a_2 + b_2\sqrt{2})$

8 Space of Polynomials

Let $\mathbb{F}$ be any field, and k be a sub-field of F. Define

$$F[x] = \{a_0 + a_1 t + a_2 t^2 + \dots\} \quad \text{where } a_0, a_1, a_2, \dots \in F \quad \text{and all } a_i = 0 \text{ except finitely many.}$$

For $(1 + x)$,

$f(t) = 1 + x + a_2 x^2 + a_3 x^3 + \dots \quad$ where $\; a_i = 0$ and $i = 1, 2, 3 \dots.$

Polynomial is eventually zero sequences. $\quad (a_1, a_2, a_3, \dots, a_n)$

Then $F[x]$ is a vector space over k, w.r.t:

1. **Vector Addition:**

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$\implies f(x) + g(x) = (a_0 + b_0)x + (a_1 + b_1)x^2 + (a_2 + b_2)x^2 + \dots$$

2. **Scalar Multiplication:**

$$\alpha f(x) = \alpha a_0 x + \alpha a_1 x^2 + \alpha a_2 x^3 + \dots$$

3. **Zero Vector:** Zero Polynomial

$$O(x) = 0 + 0x + 0x^2 + 0x^3 + \dots$$

Note: Zero polynomial of degree is not defined. Some say that its degree is -1.]

Examples

- $F = \mathbb{R},\ k = \mathbb{R}$
  - $\mathbb{R}(x)(\mathbb{R})$ is a Vector Space
- $F = \mathbb{R},\ k = \mathbb{Q}$
  - $\mathbb{R}(x)(\mathbb{Q})$ is a Vector Space
- $F = \mathbb{R},\ k = \mathbb{Q}(\sqrt{2})$
  - $\mathbb{R}(x)(\mathbb{Q}(\sqrt{2}))$ is a Vector Space.
- $F = \mathbb{Q},\ k = \mathbb{Q}$
  - $\mathbb{Q}(x)(\mathbb{Q})$ is a Vector Space
- $F = \mathbb{Q}[\sqrt{2}],\ k = \mathbb{Q}$
  - $\mathbb{Q}\sqrt{2})(x)(\mathbb{Q})$ is a Vector Space.
- $F = \mathbb{C}\ \ k = \mathbb{R}$
  - $\mathbb{C}(x)(\mathbb{R})$ is a Vector Space.
    Also,
  - $\mathbb{C}(x)(\mathbb{Q})$ is a Vector Space
  - $\mathbb{C}(x)(\mathbb{Q}\sqrt{2})$ is a Vector Space.
  - $\mathbb{C}(x)(\mathbb{Q})(i)$ is a Vector Space.
  - $\mathbb{C}(x)(\mathbb{C})$ is a Vector Space.

9 Space of Functions

Let F be any field, and k be a sub-field of F. Suppose $S \neq \phi$. Define

$$F^S = \{f \mid f : S \to F\} \quad \text{(collection of all functions from } S \text{ to } F\text{)}.$$

Then F^S is a vector space over k, w.r.t:

1. **Vector Addition:** For $f, g \in \mathbb{V}$,
$$(f+g)(s) = f(s) + g(s) \quad \forall\, s \in S$$
(Addition $f+g$ is field addition).
2. **Scalar Multiplication:** For $\alpha \in k$, $f \in V$,
$$(\alpha f)(s) = \alpha f(s) \quad \forall\, s \in S$$
3. **Zero Vector:** Zero function:
$$O : S \to F \quad \text{such that} \quad O(s) = 0 \quad \forall\, s \in S$$
$$f(0)(s) = f(s) + 0(s) = f(s) + 0 = f(s) \implies f + 0 = f$$

Example: Let $S = \{u, v\}$, $F = \mathbb{Z}_2 = \{0, 1\}$.

$$F^S = \{f \mid f : S \to \mathbb{Z}_2\}$$
$$F^S = \{O, f, g, h\}$$
$$f(u) = 0,\ f(v) = 1,$$
$$g(u) = 1,\ g(v) = 0$$
$$h(u) = 1,\ h(v) = 1$$

$\mathbb{F}^S(\mathbb{F})$ is a vector space.

Note: Two functions f and g are equal if their domain and co-domain are equal, also their value on each point is also equal.

10 Space of Sequence

Let F be any field, and k be a sub-field of F. Define

$$V = F^{\infty} = \{\langle a_n \rangle \mid a_n \in F, \ \forall \ n \in \mathbb{N}\}$$

$$\langle a_n \rangle = \langle a_1, a_2, a_3, a_4, a_5, \dots \rangle$$

Then $\mathbb{V}$ is a vector space over k w.r.t:

1. **Vector Addition:**
$$\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$$
2. **Scalar Multiplication:**
$$\alpha \cdot \langle a_n \rangle = \langle \alpha a_n \rangle \qquad \forall \ \alpha \in K$$
3. **Zero Vector:**
$$\langle 0 \rangle = \langle 0, 0, 0, 0, 0, \dots \rangle$$

$$\mathbf{F} = \mathbb{R}, \quad \mathbf{k} = \mathbb{R}$$

$$F^{\infty} = \{\langle a_n \rangle = \langle a_1, a_2, a_3, a_4, \dots \rangle \mid a_1, a_2, \dots \in \mathbb{R}\}$$

Examples:

$$\langle 1, 1, 1, 1, 1, \dots \rangle$$
$$\langle 2, 2, 1, 2, 1, 2, \dots \rangle$$
$$\langle \frac{1}{2}, \frac{1}{2}, \dots \rangle$$
$$\langle 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle$$
$$\langle 0, 0, 0, 0, 0, \dots \rangle$$

Operations:

$$\langle a_n \rangle = \langle a_1, a_2, a_3, \dots \rangle$$
$$\langle b_n \rangle = \langle b_1, b_2, b_3, \dots \rangle$$
$$\langle a_n \rangle + \langle b_n \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots \rangle$$
$$\alpha \langle a_n \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3, \dots \rangle$$

Example with zero vector:

$$\begin{aligned} \langle a_n \rangle + \langle 0 \rangle &= \langle a_1, a_2, a_3, \dots \rangle + \langle 0, 0, 0, \dots \rangle \\ &= \langle a_1 + 0, a_2 + 0, a_3 + 0, \dots \rangle \\ &= \langle a_1, a_2, a_3, \dots \rangle \end{aligned}$$

11 Properties of Subspaces

#1. Let $\mathbb{V}(F)$ be a vector space, then $W = \{0\}$ is a subspace of V.

1. $0 \in V$
2. Let any $x, y \in W$, so $x = y = 0$
$$\Rightarrow x - y = 0 - 0 = 0 \in W$$
3. $\alpha \in F, \ x \in W \Rightarrow \alpha \cdot 0 = 0 \in W$

#2. If $\mathbb{V}(F)$ is a vector space, then V is a subspace of itself.

> **Note:** $\{0\}$ is called the **trivial subspace** of V.

1. Any subspace other than $\{0\}$ is called a **non-trivial subspace**.

2. If W is a subspace of V and $W \neq V$, then W is called a **proper subspace** of V.

$$V = \mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$$
$$W = \{(x,0) \mid x \in \mathbb{R}\} \neq \{(0,0)\}$$

Then W is a subspace of V. Also, $W \neq V \Rightarrow W$ is a proper subspace of V.

#3. If $W_1, W_2, \ldots, W_n$ are subspaces of $V(F)$, then

$$W_1 \cap W_2 \cap W_3 \cap \cdots \cap W_n$$

is also a subspace of $\mathbb{V}$.

Proof: Let $x, y \in W_1 \cap W_2 \cap \cdots \cap W_n$ and $\alpha, \beta \in F$.

$$\Rightarrow x, y \in W_i \quad \forall i = 1, 2, 3, \ldots, n$$
$$\Rightarrow \alpha x + \beta y \in W_i \quad \forall i = 1, 2, \ldots, n \text{ (since } W_i \text{ is a subspace)}$$
$$\Rightarrow \alpha x + \beta y \in W_1 \cap W_2 \cap \cdots \cap W_n$$
$$\Rightarrow W_1 \cap W_2 \cap \cdots \cap W_n \text{ is a subspace.}$$

#4. Union of two subspaces need not be a subspace.

Example: Let $V = \mathbb{R}^2,\ F = \mathbb{R}$.

$$W_1 = \{(x,0) \mid x \in \mathbb{R}\}$$
$$W_2 = \{(0,y) \mid y \in \mathbb{R}\}$$

Then W_1 and W_2 both are subspaces. But $W_1 \cup W_2$ need not be a subspace of V.

$$x = (1,0) \in W_1 \subseteq W_1 \cup W_2$$
$$y = (0,1) \in W_2 \subseteq W_1 \cup W_2$$

Now

$$x + y = (1,-1) \notin W_1,\ \notin W_2 \quad \notin W_1 \cup W_2$$

#5. Union of two subspaces is a subspace if and only if one of them is contained in the other.

Proof:

$$\Rightarrow \text{Let} \mathbb{V}(F) \text{be a vector space.}$$
$$\Rightarrow \text{Let } W_1, W_2 \text{ be subspaces of } V.$$

To show $W_1 \cup W_2$ is a subspace of $\mathbb{V}$,

$$\Leftrightarrow \textbf{ either } W_1 \subseteq W_2 \textbf{ or } W_2 \subseteq W_1.$$

Let $W_1 \cup W_2$ be a subspace.

To show: $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Let if, neither W_1 is contained in W_2, i.e., $W_1 \nsubseteq W_2$, nor $W_2 \subseteq W_1$.

$$\Rightarrow \exists\, x \in W_1 : x \notin W_2$$
$$\Rightarrow \exists\, y \in W_2 : y \notin W_1$$

Now,

$$x \in W_1 \subseteq W_1 \cup W_2$$
$$y \in W_2 \subset W_1 \cup W_2$$
$$x, y \in W_1 \cup W_2$$

$$\implies x + y \in W_1 \cup W_2 \quad \text{(since } W_1 \cup W_2 \text{ is a subspace)}$$
$$\implies x + y \in W_1 \quad \text{or } x + y \in W_2$$
$$\implies x + y \in W_1, \text{ or } x + y \in W_2$$
$$(x+y) - x \in W_1 \quad \text{or } (x+y) - y \in W_2$$

$$y \in W_1 \quad \text{and } x \in W_2$$

This is a contradiction on sign of Shukla Sir.

$$\therefore \text{Either } W_1 \subseteq W_2 \text{ or } W_2 \subseteq W_1.$$

Converse Part :

Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

To show: $W_1 \cup W_2$ is a subspace.

$$\text{since } W_1 \subseteq W_2 \quad \Rightarrow W_1 \cup W_2 = W_2 \quad (\text{subspace of } V).$$

& if

$$W_2 \subseteq W_1 \quad \Rightarrow W_2 \cup W_1 = W_1 \quad (\text{subspace of } V).$$

Question. Let $V = \mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \; : \; a_1, a_2, \ldots, a_n \in \mathbb{R}\}$ considered as a vector space over $\mathbb{R}$. Then which of the following is (are) subspace(s) of V?

(A). $W_1 = \{(a_1, a_2, a_3) \in V \; : \; a_1 = 2a_2 + 3a_3\}$

(i) $0 = \{(0, 0, 0, 0, 0, \ldots)\} \quad \Rightarrow \; a_1, a_2, a_3$

$$a_1 = 2a_2 + 3a_3 \quad \Rightarrow 0 = 2 \cdot 0 + 3 \cdot 0$$
$$\Rightarrow 0 \in W.$$

(ii)

$$x = (x_1, x_2, x_3 \cdots, x_n) \implies x_1 = 2x_2 + 3x_3$$
$$y = (y_1, y_2, y_3 \cdots, y_n) \implies y_1 = 2y_2 + 3y_3$$

$$x - y = x_1 - y_1, x_2 - y_2, \cdots x_n - y_n \implies x_1 - y_1 = 2(x_2 - y_2) + 3(x_3 - y_3)$$

$x - y = (F_{\nVdash}, F_{\nvDash} \cdots F_{\ltimes}) \implies F_{\nVdash} = 2F_{\nvDash}$

For subspace:

$$0_v \in W, \quad x - y \in W$$
$$\forall \alpha \in \mathbb{F}, \; \alpha x \in W \quad \text{where } W \text{ satisfies.}$$
$$\alpha \in \mathbb{R}, \; x = (x_1, x_2, x_3 \ldots x_n) \text{ and}$$
$$x_1 = 2x_2 + 3x_3$$
$$\Rightarrow \alpha x_1 = 2\alpha x_2 + 3\alpha x_3$$
$$\Rightarrow x_1 = 2x_2 + 3x_3$$

$$\therefore W_1 \text{ is a subspace of } \mathbb{V}.$$

12 Subspaces in Vector Spaces

Let V be a vector space over a field $\mathbb{F}$, and let W_1 and W_2 be subspaces of V.

$$x + y \text{ or } x - y \in w_1 \cup w_2$$

Let $\mathbf{x}, \mathbf{y} \in W_1 \cap W_2$:

$$\mathbf{x} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \in W_1 \subseteq W_1 \cup W_2,$$

$$\mathbf{y} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \in W_2, \mathbf{y} \in W_1, W_2.$$

$$\sum_{j=1}^{n} a_{mj} = 0.$$

Let

$$W_2 = \left\{ (a_{ij}) \in V \;\middle|\; \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = 0 \right\}.$$

For $j = 1, 2, \ldots, n$, we have:

Then, W_1 and W_2 are subspaces over $V = \mathbb{F}^{m \times n}$.

12.1 Conditions

i) $W_1 \subseteq W_2$
ii) $W_2 \subseteq W_1$
iii) $W_1 \cap W_2 = \{\mathbf{0}\}$

12.2 Sum of Two Subspaces

Let $V(F)$ be any vector space, and suppose W_1 and W_2 are subspaces of V.

Then the set ,

$$W_1 + W_2 = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in W_1, \mathbf{y} \in W_2\}.$$

This is called the **sum of W_1 and W_2**.

12.3 Theorem

Let $V(F)$ be any vector space, and let W_1, W_2 be subspaces of V.

i) $W_1 + W_2$ is a subspace of V.
ii) $W_1 \subseteq W_1 + W_2$, $W_2 \subseteq W_1 + W_2$.
iii) If W is a subspace of V such that $W_1 \subseteq W$ and $W_2 \subseteq W$,

then $W_1 + W_2 \subseteq W$.

From this, we can say that $W_1 + W_2$ is the smallest subspace which contains both the subspaces W_1 and W_2.

Thus, $W_1 + W_2$ is the **smallest subspace** containing both W_1 and W_2.

Proof:

1.

$$W_1 + W_2 = \{x + y \mid x \in W_1, y \in W_2\}$$

(i) $0 \in W_1$, $0 \in W_2$

$$0 = 0 + 0 \in W_1 + W_2 \implies 0 \in W_1 + W_2$$

(ii) Let $\alpha, \beta \in \mathbb{F}$, $\quad u \in W_1 + W_2$.

To show: $\alpha u + \beta v \in W_1 + W_2$.

Now

$$u = x_1 + y_1,\ x_1 \in W_1,\ y_1 \in W_2 \quad \text{and} \quad v = x_2 + y_2,\ x_2 \in W_1,\ y_2 \in W_2.$$

Now,

$$\alpha u + \beta v = \alpha(x_1 + y_1) + \beta(x_2 + y_2) \implies \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2$$

Since

$$\alpha u + \beta v \in W_1 + W_2 \implies W_1 + W_2 \text{ is a subspace.}$$

(2.) Let $x \in W_1$

$$x = x(\in W_1) + 0(\in W_2) \in W_1 + W_2,$$
$$x \in W_1 + W_2 \implies \boxed{W_1 \subseteq W_1 + W_2}.$$

Let $y \in W_2$

$$0 + y = y \in W_1 + W_2,$$
$$y \in W_1 + W_2 \implies \boxed{W_2 \subseteq W_1 + W_2}.$$

(3.) Let W be a subspace of V and $W_1, W_2 \subseteq W$. Then let

$$z \in W_1 + W_2,$$
$$z = x + y,\ x \in W_1,\ y \in W_2.$$

Now

$$x \in W_1 \subseteq W \implies x \in W, \quad y \in W_2 \subseteq W \implies y \in W,$$

$$z = x + y \implies z \in W.$$

$$\boxed{W_1 + W_2 \subseteq W}.$$

$W_1 + W_2$ (smallest subspace)	W_1, W_2 (subspaces of W)

Ex: $V = \mathbb{R}^{2*2}$

$$W_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\},$$
$$W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$$

$$W_1 + W_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$$

13 Direct Sum of Two Subspaces

Let $V(\mathbb{F})$ be any vector space, and let W_1, W_2 be two subspaces of V. Then we say $'V'$ is a **direct sum** of W_1 and W_2, and we write:

$$V = W_1 \oplus W_2$$

If: (i) $V = W_1 + W_2$ (i.e., $\forall x \in V \implies x = u + v, u \in W_1, w \in W_2$)

(ii) $W_1 \cap W_2 = \{0\}$

Ex: Let $V = \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$

$$W_1 = \{(a, 0) \mid a \in \mathbb{R}\},$$
$$W_2 = \{(0, b) \mid b \in \mathbb{R}\}$$

Now,

$$\forall\ (x, y) \in \mathbb{R}^2,$$
$$\implies (x, y) = (x, 0) + (0, y), \qquad \text{where } (x, 0) \in W_1 \text{ and} (0, y) \in W_2$$
$$\boxed{\implies \mathbb{R}^2 = W_1 + W_2}$$

Now,

$$(x, y) \in W_1 \cap W_2$$
$$\implies (x, y) \in W_1\ \&\ (x, y) \in W_2$$
$$\implies y = 0\ \&\ x = 0$$
$$\implies (x, y) = (0, 0)$$
$$\implies W_1 \cap W_2 = \{(0, 0)\}$$
$$\implies \mathbb{R}^2 = W_1 \oplus W_2$$

Definition:

Let $P_n(\mathbb{F})$ denote the space of all polynomials of degree at most n over $\mathbb{F}$. i.e.,

$$P_n(f) = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{F}\}$$

Ex:

$$P_1(\mathbb{R}) = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$$
$$P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

Ex:

$$P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$
$$W_1 = \{c + dx^2 \mid c, d \in \mathbb{R}\},$$
$$W_2 = \{ex \mid e \in \mathbb{R}\}$$

13.1 Verify (i) W_1 and W_2 are subspaces (Home-work):

$$\forall\ V \in P_2(\mathbb{R})$$
$$\implies V = a_0 + a_1x + a_2x^2$$
$$V = (a_0 + a_2x^2) + a_1x$$

$$\implies P_2\mathbb{R} = W_1 + W_2$$

$$a_0 + a_1x + a_2x^2 \in W_1 \cap W_2$$
$$\implies a_0 + a_1x + a_2x^2 \in W_1 \ \&\ a_0 + a_1x + a_2x^2 \in W_2$$
$$\implies a_1 = 0 \ \&\ a_0 = 0, a_2 = 0$$
$$\implies a_0 = a_1 = a_2 = 0$$
$$\implies a_0 + a_1x + a_2x^2 = 0 + 0.x + 0.x^2$$

$$\boxed{\implies W_1 \cap W_2 = \{0\} = P_2(\mathbb{R}) = W_1 \oplus W_2}$$

$$W_1 = \{c + dx^2 \mid c, d \in \mathbb{R}\}$$

$$W_2 = \{ex \mid e \in \mathbb{R}\}$$

Let

$$x_1 = a + bx^2 \quad \text{and} \quad y = c + dx^2$$
$$\alpha, \beta \in \mathbb{F}$$
$$\alpha x + \beta y = \alpha(a + bx^2) + \beta(c + dx^2)$$
$$= (\alpha a + \beta c) + (\alpha b + \beta d)x^2 \in W_1$$
$$\implies W_1 \text{ is a subspace of } P_2(\mathbb{R})$$

$$W_2 = \{ex \mid e \in \mathbb{R}\}$$
$$x = ex \quad \text{and} \quad y = fx$$
$$\alpha, \beta \in \mathbb{F}$$
$$\alpha x + \beta y = \alpha ex + \beta ex$$
$$= (e\alpha + \beta e)x \in W_2$$
$$\implies W_2 \text{ is a subspace of } P_2(\mathbb{R})$$

Theorem:

Let $V(F)$ be a V.S. and W_1, W_2 be subspaces of V.

$$\text{Then } V = W_1 \oplus W_2 \iff \forall x \in V,\ x = u + v, \text{ for unique vectors } u \in W_1,\ v \in W_2.$$

Example:

$$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$$

$$W_1 = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$$

$$W_2 = \{(0, x_2) \mid x_2 \in \mathbb{R}\}$$

$$\mathbb{R}^2 = W_1 \oplus W_2$$
$$\forall (x_1, x_2) \in \mathbb{R}^2$$
$$(x_1, x_2) = (x_1, 0) + (0, x_2)$$

$$(1, 3) = (1, 0) + (0, 3)$$

Definition: For Direct Sum:

$$V = W_1 \oplus W_2 \oplus W_3$$

$\iff$

$$\begin{aligned}&\textbf{i.}W_1, W_2, W_3 \text{ are subspaces of } V\\&\textbf{ii.}V = W_1 + W_2 + W_3,\ \text{i.e., } \forall x \in V,\ x = u_1 + u_2 + u_3,\ u_i \in W_i,\ i = 1, 2, 3\\&\textbf{iii.}W_1 \cap W_2 = \{0\}, \qquad (W_1 + W_2) \cap W_3 = \{0\}\end{aligned}$$

$$V = W_1 \oplus W_2 \oplus W_3 \oplus \cdots \oplus W_n$$

$$\textbf{i.}W_1, W_2, \ldots, W_n \text{ are subspaces of } V$$
$$\textbf{ii.}V = W_1 + W_2 + \cdots + W_n$$
$$\textbf{iii.}(W_1 + W_2 + \cdots + W_{i-1}) \cap W_i = \{0\},\ \forall i = 1, 2, 3, \ldots, n$$

14 Quotient Space

Let $V(F)$ be any vector space, and let W be a subspace of V. Define a relation "$\sim$" on V as:

For $a, b \in V$,

$$\boxed{a \sim b \iff a - b \in W}$$

Then, $\sim$ is an equivalence relation.

14.1 Equivalence Classes

$$\bar{x} = \{y \in V \mid y \sim x\} = \{y \in V \mid y - x \in W\}$$

Equivalence class of x:

$$\bar{x} = \{y \in V \mid y - x = w,\ w \in W\} = \{y = a + w,\ w \in W\}$$
$$\bar{x} = \{x + W \mid w \in W\} \qquad (*)$$
$$\boxed{\bar{x} = x + W}$$

Now:

$$\begin{aligned}\bar{a} = \bar{b} &\iff a \sim b\\&\iff a - b \in W\\\textbf{i.e. } a + W &= b + W\\&\implies a - b \in W\end{aligned}$$

Note:

$$\begin{aligned}&\bar{a} = \bar{0}\\&\iff a + W = 0 + W = W\\&\iff a \sim 0\\&\iff a - 0 \in W\\&\iff a \in W \qquad (*)\end{aligned}$$

Now, let $V(F)$ be any vector space and W be a subspace of V. Then the set

$$\frac{V}{W} = \{\bar{x} = x + W \mid x \in V\}$$

is a vector space over F.

14.2 Operations in $\frac{V}{W}$:

1. **Vector addition:**

$$\bar{x} = x + w,\ \bar{y} = y + w$$
$$\boxed{\bar{x} + \bar{y} = (x + y) + W}$$

2. **Scalar multiplication:**

$$\alpha \cdot \bar{x} = \alpha(x + W) =$$

$$\boxed{(\alpha x) + W}$$

3. **Zero vector**:

$$\bar{0} = 0 + W$$
$$\bar{x} + \bar{0} = (x + 0) + W$$
$$\implies x + W = \bar{x}$$
$$\forall \bar{x} \in \frac{V}{W}$$
$$\boxed{,\ \bar{0} = 0 + W = W}$$

Thus, $0 + W$ is the zero vector.

15 Linear Combination

Let $V(F)$ be a vector space and W be a subspace, and

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k \in V \quad \text{and} \quad \alpha_1, \alpha_2, \ldots, \alpha_k \in F.$$

Then,

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_k \mathbf{x}_k \in V$$

is called a **linear combination** of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.
x is the Linear Combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k$

Example 1:

Let $V = \mathbb{R}^3$, $F = \mathbb{R}$. Consider,

$$\mathbf{x}_1 = (1, 0, 0), \quad \mathbf{x}_2 = (0, 1, 1), \quad \mathbf{x}_3 = (0, 0, 1).$$

Let $\mathbf{u} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3$. Then,

$$\begin{aligned} &= \alpha_1(1,0,0) + \alpha_2(0,1,1) + \alpha_3(0,0,1) \\ &= (\alpha_1, 0, 0) + (0, \alpha_2, \alpha_2) + (0, 0, \alpha_3) \\ &= (\alpha_1, \alpha_2, \alpha_2 + \alpha_3). \end{aligned}$$

For $\alpha_1 = 1$, $\alpha_2 = 0$,$\alpha_3 = 1$, we get:

$$(1, 0, 0 + 1) = (1, 0, 1).$$

Thus, $\mathbf{x}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

Example 2:

Let $V = \mathbb{R}^3$, $F = \mathbb{R}$. Consider,

$$\mathbf{d}_1 = (1, 0, 1), \quad \mathbf{d}_2 = (0, 1, 1).$$

Let $\mathbf{u} = (1, 2, 3)$. Check if $\mathbf{u}$ is a linear combination of $\mathbf{d}_1$ and $\mathbf{d}_2$. Assume:

$$\mathbf{u} = \alpha_1 \mathbf{d}_1 + \alpha_2 \mathbf{d}_2.$$

Then,

$$\begin{aligned} (1, 2, 3) &= \alpha_1(1, 0, 1) + \alpha_2(0, 1, 1) \\ &= (\alpha_1, 0, \alpha_1) + (0, \alpha_2, \alpha_2) \\ &= (\alpha_1, \alpha_2, \alpha_1 + \alpha_2). \end{aligned}$$

Equating components:

$$\begin{aligned} \alpha_1 &= 1, \\ \alpha_2 &= 2, \\ \alpha_1 + \alpha_2 &= 3. \end{aligned}$$

However, $1 + 3 \neq 2$. Hence, $\mathbf{u}$ is **not** a linear combination of $\mathbf{d}_1$ and $\mathbf{d}_2$.

–

16 Linear Span

Let $V(F)$ be a vector space, and let S be any non-empty subset of V. Then we define the *linear span* of S by $L(S)$ or $\text{Span}(S)$ or $\langle S \rangle$. It is defined as:

$$L(S) = \{\mathbf{u} \mid \mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k,\ \alpha_i \in F, \mathbf{v}_i \in S, \text{and } k \text{ is finite}\}.$$

If $S = \emptyset$, then $L(S) = \{\mathbf{0}\}$.

$$L(S) = \text{Set of all linear combinations of elements of } S.$$

16.1 Examples.

Example 1

Let $S = \{(1, 1, 0), (0, 1, 1)\} \subseteq \mathbb{R}^3$, $F = \mathbb{R}$. Then,

$$\begin{aligned} L(S) &= \{\alpha_1(1, 1, 0) + \alpha_2(0, 1, 1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\ &= \{(\alpha_1, \alpha_1, 0 + 0, \alpha_2, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}. \qquad = \{(\alpha_1, \alpha_1 + \alpha_2, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}. \end{aligned}$$

Thus, $L(S)$ is the set of all linear combinations of $\{(1, 1, 0), (0, 1, 1)\}$.

Example 2:

Let $V = \mathbb{R}^{2\times 3}$, $F = \mathbb{R}$. Consider,

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Find $L(S)$.

Solution:

We have,

$$L(S) = \left\{ \alpha_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}.$$

Expanding, we get:

$$L(S) = \left\{ \begin{pmatrix} \alpha_1 + \alpha_2 & \alpha_2 + \alpha_3 & \alpha_3 \\ \alpha_1 & \alpha_1 & \alpha_2 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}.$$

Thus, $L(S)$ is the set of all matrices of the form

$$\begin{pmatrix} \alpha_1 + \alpha_2 & \alpha_2 + \alpha_3 & \alpha_3 \\ \alpha_1 & \alpha_1 & \alpha_2 \end{pmatrix},$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

16.2 Properties of Linear Span

1.Let $V(F)$ be any vector space and $S \subseteq V$. Then:

1. $L(S)$ is a subspace of V.
2. $S \subseteq L(S)$.
3. If W is a subspace of V such that $S \subseteq W$, then $L(S) \subseteq W$.

> L(S) is the smallest subspace of V that contains S.

2. Let $L(S) = S$. Then S is a subspace of V.

Proof:
Forward: Let $L(S) = S$. Then S is a subspace.

Converse: Let S be a subspace. Also, $S \subseteq S$. Thus, $L(S) = S$.

3. If $S_1, S_2 \subseteq V$, then:

$$L(S_1 \cup S_2) = L(S_1) + L(S_2).$$

4. $S_1 \subseteq S_2 \subseteq V$ then :

$$L(S_1) \subseteq L(S_2)$$

5. L(L(S))=L(S)

Illustration:

Let $V = \mathbb{R}^2$,

$$S_1 = \{(1,0)\},$$
$$S_2 = \{(0,1)\}.$$

Then,

$$L(S_1) = \{\alpha_1(1,0) \mid \alpha_1 \in \mathbb{R}\} = \{(\alpha_1, 0) \mid \alpha_1 \in \mathbb{R}\},$$
$$L(S_2) = \{\alpha_2(0,1) \mid \alpha_2 \in \mathbb{R}\} = \{(0, \alpha_2) \mid \alpha_2 \in \mathbb{R}\},$$
$$L(S_1 \cup S_2) = \{\alpha_1(1,0) + \alpha_2(0,1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$
$$= \boxed{\{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}.}$$

Also,

$$L(S_1) + L(S_2) = \{(\alpha_1, 0) + (0, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$
$$= \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}.$$

Linearly Dependent Set

Let $V(F)$ be a vector space, and $S \subseteq V$, we say that S is linearly dependent over F

If there exist $u_1, u_2, \dots, u_k \in S$
and $\alpha_1, \alpha_2, \dots, \alpha_k \in F$
(not all zero) i.e. $\exists$ at least one $\alpha_i \neq 0$

such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = \mathbf{0} \quad \text{(zero vector)}$$

i.e., $\exists\, u_1, u_2, \dots, u_k \in S$
such that the equation

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = \mathbf{0}$$

has at least one non-zero solution.

Example. Let $V = \mathbb{R}^3(\mathbb{R})$,
$S = \{(1,1,2),(2,4,2),(3,0,1)\}$

$u_1 = (1,2,1)$,
$u_2 = (2,4,2)$,
$\alpha_1 = -2 \neq 0$,
$\alpha_2 = 1 \neq 0$

$$-2 \cdot (1,2,1) + 1 \cdot (2,4,2) = (-2,-4,-2) + (2,4,2) = (0,0,0)$$

$\implies$ S is linearly dependent.

Check whether $S = \{(1,2,1),(2,4,2),(3,0,1)\}$ is L.D.?

Solution:

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \mathbf{0} \quad \implies \quad (0,0,0)$$

Consider matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Row reduce:

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{-6}, \quad R_3 \to \frac{R_3}{-2}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Let } x_1 = 2x_2, \quad x_2 = x_3$$

$$\implies \alpha_1 = 2\alpha_2, \quad \alpha_2 + \alpha_3 = 0$$

Thus, S is linearly dependent.

17 Linear Independence and Dependence

Let $V(F)$ be any vector space, and $S \subseteq V(F)$, then
S is said to be linearly independent if S is not linearly dependent.

$$\text{i.e. } \forall\, u_1, u_2, \ldots, u_k \in S, \quad \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = 0$$
$$\implies \alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = 0$$

17.1 Linear dependence and linear independence of finite set

Let $V(F)$ be any vector space and

$$S = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq V$$

Then S is linearly independent if:

$$\iff \boxed{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = 0}$$
$$\boxed{\implies \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0}$$

Example

Let $V = \mathbb{R}_3(\mathbb{R})$

$$S = \{x + 1, x + x^2, x^2 + x^3\}$$

Consider:

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0 \implies 0 + 0x + 0x^2 + 0x^3$$

$$\implies \alpha_1(x + 1) + \alpha_2(x + x^2) + \alpha_3(x^2 + x^3) = 0 + 0x + 0x^2 + 0x^3$$

$$\alpha_1 + (\alpha_2 + \alpha_3)x^2 + \alpha_3 x^3) = 0 + 0x + 0x^2 + 0x^3$$

$$\implies \alpha_1 + \alpha_2 + 2\alpha_3 = 0, \quad \alpha_2 t + \alpha_3 x^2 = 0x + 0x^2$$

$$\alpha_1 = 0, \quad \alpha_1 + \alpha_2 = 0, \quad +\alpha_2 + \alpha_3 = 0$$
$$\boxed{\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0}$$
$$\therefore S \text{ is linearly independent.}$$