Special Theory of Relativity in Electrodynamics

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1 Postulates of Special Relativity

In 1905, Einstein in his paper stated two postulates of special theory of relativity. These are as follow:

- **Postulate 1**: The principle of relativity: the laws of physics are the same in all inertial frames.
- **Postulate 2**: The speed of light in vacuum is the same in all inertial frames

In addition, we will assume that the stage our physical laws act on is homogeneous and isotropic. This means it does not matter where (=homogeneity) we perform an experiment and how it is oriented (=isotropy), the laws of physics stay the same.

2 The Invariant of Special Relativity

Let's start with a thought experiment that enables us to derive one of the most fundamental consequences of the postulates of special relativity. Imagine, we have a spectator, standing at the origin of his coordinate system and sending a light pulse straight up, where it is reflected by a mirror and finally reaches again the point from where it was sent.

We have three important events:

- A : the light leaves the starting point

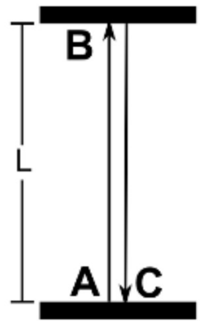

Figure 1: Illustration of the thought experiment

- B : the light is reflected at a mirror
- C : the light returns to the starting point.

The time-interval between A and C is

$$t_A - t_C = \Delta t = \frac{2L}{c}$$

Next imagine a second spectator, standing at t_A at the origin of his coordinate system and moving with constant velocity u to the left, relative to the first spectator. The second spectator sees things a little differently. In his frame of reference the point where the light ends up will not have the same coordinates as the starting point.

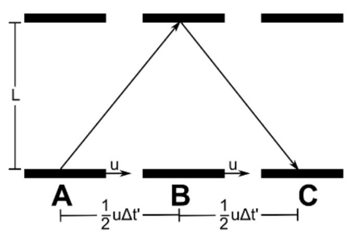

Figure 2: Illustration of the thought experiment for a moving spectator

$$x'_A = 0 \neq x'_C = u\Delta t' \text{ß} \Delta x' = u\Delta t'$$

The time interval $\Delta t' = t'_C - t'_A$ is equal to the distance l the light travels, as the second spectator observes it, divided by the speed of light c.

$$\Delta t' = \frac{l}{c}$$

We can compute the distance traveled l using the Pythagorean theorem

$$l = 2\sqrt{\left(\frac{1}{2}u\Delta t'\right)^2 + L^2}$$

Therefore,

$$c\Delta t' = 2\sqrt{\left(\frac{1}{2}u\Delta t'\right)^2 + L^2}$$

Hence,

$$\Delta x' = u\Delta t'$$

$$c\Delta t' = 2\sqrt{\left(\frac{1}{2}\Delta x'\right)^2 + L^2}$$

$$(c\Delta t')^2 = 4\left(\left(\frac{1}{2}\Delta x'\right)^2 + L^2\right)^2$$

$$(c\Delta t')^2 - (\Delta x')^2 = 4\left(\left(\frac{1}{2}\Delta x'\right)^2 + L^2\right)^2 - (\Delta x')^2 = 4L^2$$

Therefore, we have found something which is the same for all observers: the quadratic form

$$\boxed{(\Delta s)^2 \equiv (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$$

3 The Minkowski Notation

We can rewrite the invariant of special relativity as

$$ds^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

we can write above equation as,

$$ds^2 = \eta^{\mu\nu} dx_\mu dx_\nu = \eta^{00}(dx_0)^2 + \eta^{11}(dx_1)^2 + \eta^{22}(dx_2)^2 + \eta^{33}(dx_3)^2$$

Here Minkowski metric $\eta^{00} = 1, \eta^{11} = -1, \eta^{22} = -1, \eta^{33} = -1,$ and $\eta^{\mu\nu} = 0$ (an equal way of writing this is $\eta = \text{diag}(1, -1, -1, -1)$)

4 Einstein summation convention

If an index occurs twice, a sum is implicitly assumed :

$$\sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

but

$$\sum_{i=1}^{3} a_i b_j = a_1 b_j + a_2 b_j + a_3 b_j \neq a_i b_j$$

5 Four-Vectors

Renaming of the variables $x_0 = ct, x_1 = x, x_2 = y$ and $x_3 = z$, to make it obvious that time and space are now treated equally and to be able to use the rules introduced above. In addition, it's conventional to introduce the notion of a four- vector

$$dx_\mu = \begin{pmatrix} dx_0 \\ dx_1 \\ dx_2 \\ dx_3 \end{pmatrix},$$

because the equation above can be written equally using four-vectors and the Minkowski metric in matrix form

$$(ds)^2 = dx_\mu \eta^{\mu\nu} dx_\nu = \begin{pmatrix} dx_0 & dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$$

A physical interpretation of ds is that it is the "distance" between two events in spacetime. The mathematical tool that tells us the distance between two infinitesimal separated points is called metric. Length of a four-vector, which is given by the scalar product of the vector with itself

$$x^2 = x \cdot x = x_\mu x_\nu \eta^{\mu\nu}$$

Analogously, the scalar product of two arbitrary four-vectors is given by,

$$x \cdot y = x_\mu y_\nu \eta^{\mu\nu}$$

There is another, notational convention to make computations more streamlined. We define a four-vector with upper index as

$$x^\mu = \eta^{\mu\nu} x_\nu$$

The Minkowski metric is symmetric $\eta^{\mu\nu} = \eta^{\nu\mu}$

6 Lorentz Transformation

It follows directly from the two postulates that $ds^2 = \eta^{\mu\nu} dx_\mu dx_\nu$ is the same in all inertial frames of reference:

$$ds'^2 = dx'_\mu dx'_\nu \eta^{\mu\nu} = ds^2 = dx_\mu dx_\nu \eta^{\mu\nu}$$

Therefore, allowed transformations are those which leave this quadratic form or equally the scalar product of Minkowski spacetime invariant. Denoting a generic transformation that transforms the description in one frame of reference into the description in another frame with Λ, the transformed coordinates dx'_μ can be written as:

$$dx_\mu \to dx'_\mu = \Lambda^\sigma_\mu dx_\sigma$$

Then we can write the invariance condition as

$$(ds)^2 = (ds')^2$$

$$dx \cdot dx = dx' \cdot dx'$$

$$dx_\mu dx_\nu \eta^{\mu\nu} = dx' \cdot dx'$$

$$dx_\mu dx_\nu \eta^{\mu\nu} = dx'_\mu dx'_\nu \eta^{\mu\nu} = \Lambda^\sigma_\mu dx_\sigma \Lambda^\gamma_\gamma dx_\gamma \eta^{\mu\nu}$$

$$\eta^{\mu\nu} = \Lambda^\mu_\sigma \eta^{\sigma\gamma} \Lambda^\nu_\gamma$$

Or written in matrix notation

$$\eta = \Lambda^T \eta \Lambda$$

We define the Lorentz transformations as those transformations that leave the scalar product of Minkowski spacetime invariant. In physical terms this means that Lorentz transformations describe changes between frames of references that respect the postulates of special relativity.

7 Conserved Currents

We started these lectures by discussing the charge density $\rho(x,t)$, the current density $J(x,t)$ and their relation through the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$$

which tells us that charge is locally conserved. we first need to appreciate that the charge and current densities sit nicely together in a 4-vector,

$$J^\mu = \begin{pmatrix} \rho c \\ J \end{pmatrix}$$

In our new, relativistic, notation, the continuity equation takes the particularly simple form

$$\partial_\mu J^\mu = 0$$

This equation is Lorentz invariant. This follows simply because the indices are contracted in the right way: one up, and one down.

8 Gauge Potentials and the Electromagnetic Tensor

Under Lorentz transformations, electric and magnetic fields will transform into each other.

8.1 Gauge Invariance and Relativity

Let us start by recalling some of the equations of electrostatics and magnetostatics,

$$\nabla \times \mathbf{E} = 0 \implies \mathbf{E} = -\nabla \phi$$

$$\nabla \cdot \mathbf{B} = 0 \implies \mathbf{B} = \nabla \times \mathbf{A}$$

However, in general these expressions can't be correct. We know that when **B** and **E** change with time, the two source-free Maxwell equations are

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \text{ and } \nabla \cdot \mathbf{B} = 0$$

Nonetheless, it's still possible to use the scalar and vector potentials to solve both of these equations. The solutions are

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \text{ and } \mathbf{B} = \nabla \times \mathbf{A}$$

where now $\phi = \phi(x, t)$ and $\mathbf{A} = A(x, t)$ We can always shift $A \to A + \nabla \chi$ and B remains unchanged. However, now this requires a compensating shift of

$$\phi \to \phi - \frac{\partial \chi}{\partial t} \text{ and } A \to A + \nabla \chi$$

where, $\chi = \chi(x, t)$ These are gauge transformations. We define

$$A^\mu = \begin{pmatrix} \phi/c \\ A \end{pmatrix}$$

9 The Electromagnetic Tensor

From the 4-derivative $\partial_\mu = (\partial/\partial(ct), \nabla)$ and the 4-vector $A_\mu = (\phi/c, -A)$, we can form the anti-symmetric tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This is constructed to be invariant under gauge transformations.

$$F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi = F_{\mu\nu}$$

This already suggests that the components involve the E and B fields. To check that this is indeed the case, we can do a few small computations,

$$F_{01} = \frac{1}{c}\frac{\partial(-A_x)}{\partial t} - \frac{\partial(\phi/c)}{\partial x} = \frac{E_x}{c}$$

Similar computations for all other entries give us a matrix of electric and magnetic fields,

$$F_{12} = \frac{\partial(-A_y)}{\partial x} - \frac{\partial(A_x)}{\partial y} = -B_z$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_z \\ -E_x/c & -B_y & B_z & 0 \end{pmatrix}$$

$F_{\mu\nu}$ is called the electromagnetic tensor. Equivalently, we can raise both indices using the Minkowski metric to get

$$F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_z \\ E_x/c & -B_y & B_z & 0 \end{pmatrix}$$

Both $F_{\mu\nu}$ and $F^{\mu\nu}$ are tensors. They are tensors because they'reconstructed out of objects, A_μ, $\partial_{\mu\nu}$ and $\eta_{\mu\nu}$, which themselves transform nicely under the Lorentz group. This means that the field strength must transform as

$$F'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\rho F^{\rho\sigma}$$

10 Maxwell Equations

We now have the machinery to write the Maxwell equations in a way which is manifestly compatible with special relativity. They take a particularly simple form:

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^{\mu\nu}, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0}$$

The Maxwell equations are not invariant under Lorentz transformations. This is because there is the dangling ν index on both sides. Let's now check that the Maxwell equations in relativistic form do indeed coincide with the vector calculus equations that we've been studying in this course. We just need to expand the different parts of the equation.

$$\partial_i F^{i0} = \mu_0 J^0 \implies \nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\partial_\mu F^{\mu i} = \mu_0 J^i \implies -\frac{1}{c^2}\frac{\partial E}{\partial t} + \nabla \times B = \mu_0 J$$

$$\partial_i \tilde{F}^{i0} = 0 \implies \nabla \cdot \mathbf{B} = 0$$

$$\partial_\mu \tilde{F}^{\mu i} = 0 \implies \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$