Maxwell's Equations in Differential and Integral form

Shubham Yadav

Abstract

In this discussion, we explore all four **Maxwell's** equations in both **integral** and **differential** forms, and see how they describe **electric and magnetic fields**, how they interact, and why they work the way they do. We'll talk about Gauss's laws for electric and magnetic fields, which tell us about the nature of electric and magnetic flux and see some analogy of electric field with magnetic field and why there is a need of Gauss's law of Magnetic fields. Then, we will dive into Faraday's law, which explains how changing magnetic fields create electric fields, and what is the nature of that induced electric field, how differ with electrostatic field. Finally, we'll see how Ampère's Law connects electricity to magnetism and how Maxwell's correction to it led to understand the electromagnetic wave theory.

BLS01(Bose Lecture Series), Theoretical Nexus

Introduction

Maxwell's equations are the most important equations of all time, if you need a testament to the power of the equations then radio, television, Internet access and Bluetooth technology are a few examples. When Maxwell worked on his theory of electromagnetism, he ended up with not four but twenty equations that describe the behavior of electric and magnetic fields. It was ***Oliver Heaviside*** and ***Heinrich Hertz*** who combined and simplified Maxwell's Equations into four equations in the two decades after Maxwell's death.

1 Gauss's Law of Electric Fields

In Maxwell's Equations we have to deal with two types of electric fields the electrostatics fields produces by charges and the induces electric field produced a changing magnetic field.

1.1 The Integral form of Gauss's Law

Electric charges produce electric fields and the flux of that field passing to any closed surface is proportional to the total charge within the surface.

$$\oint_S \vec{E} \cdot \hat{n}\, da = \frac{Q_{enc}}{\epsilon_0} \tag{1}$$

where left side of the equation 1 is the mathematical description of electric flux. There ia an electric field density and we are integrating that density over entire surface in other words, electric flux is the number of field lines penetrating surface and right side of the equation is the total charge contained within that surface divided by a constant called permittivity of free space. $\left(\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}\right)$

Note: The unit normal($\hat{n}$) is the unit vector taken perpendicular to surface. for closed surfaced the ambiguity in the direction of unit normal has been resolved, By convention, the unit normal vector for the closed surface is taken to point outward(i.e. away from the volume closed by the surface).

- What do you think? Electric flux will define the physical motion of particle, because it's kind of linked with electric field?

  No, the electric flux is just the number of field lines that penetrate the surface, especially normal to the surface and at the same point and time electric flux may be different for two surfaces. That means they don't define the motion of a particle.

1.2 The Differential form of Gauss's Law

The divergence of electric field is equal to the volume charge density divided by the permittivity of free space.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2}$$

The divergence of electric field($\phi_{\vec{E}}$) is the tendency of field to flow away from a specified location. In other words, the electric field produced by electric charges diverges from positive charge and converges upon negative charge.

i.e. $\vec{\nabla} \cdot \vec{E} > 0$: positive divergence (electric field diverges from +Q)
$\vec{\nabla} \cdot \vec{E} < 0$: negative divergence (electric field converges upon -Q)

The key factor in determining the divergence at any point is not simply the spacing of the field lines at that point, but whether the flux out of an infinitesimal small volume around the point is greater than, equal to, or less than the flux into that volume.

- The divergence of electric field is directly proportional to the volume charge density.
- Both flux and divergence deal with the flow of a vector field but with the important difference. Flux is defined over an area while divergence applies to individual points.

Note: The problems you are most likely to encounter that can be solved using the differential form of Gauss's law involve calculating the divergence of electric field and using the result to determine the charge density at a specified location.

2 Gauss's Law of Magnetic Fields

Gauss's law of magnetic fields is similar in form but different in content from Gauss's law of electric fields. For both electric and magnetic fields, the integral form of Gauss's law involves the flux of fields over a closed surface. The key difference in the electric field and the magnetic field versions of Gauss's law arises because opposite electric charges may be isolated from one another while opposite magnetic poles (magnetic monopoles called north and south) always occur in pair i.e. no magnetic monopoles exist and the lack of isolated magnetic monopoles in nature has a profound impact on the behavior of magnetic field, and this is why the existence of Gauss's law of magnetic field is.

2.1 Integral form of the Gauss's Law

Gauss's law refers to magnetic flux which is the number of magnetic field lines passing through a closed surface. The total magnetic flux passing through any closed surface is zero. In other words, if we have a real or imaginary closed surface of any size and shape, the total magnetic flux passing through the surface must be zero.

$$\oint_S \vec{B} \cdot \hat{n}\ da = 0 \tag{3}$$

Zero flux does not mean that zero magnetic field lines penetrate the surface. This means that for every magnetic field line that enters the volume enclosed by the surface, there must be a magnetic field line leaving that volume. Thus, inward(negative)magnetic flux must be exactly equal to the outward(positive) magnetic flux.

Gauss's law for magnetic field arises directly from the lack of isolated magnetic poles (i.e. magnetic monopoles) in nature.

2.2 The Differential form of Gauss's Law

The continuous nature of magnetic field lines makes the differential form of Gauss's law quite simple. The divergence of magnetic field at any point is zero.

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4}$$

There is also another way to understand this law by analogy to electric field. For electric field, the divergence at any location is proportional to electric charge density at that location. Since it is not possible to isolate magnetic poles, so we can not have a north pole without a south pole. This is why the magnetic charge density must be zero everywhere. That means that the divergence of magnetic field must also be zero.

3 Faraday's Law

In a series of experiments in 1831, Michael Faraday demonstrated that an electric current may be induced in a circuit by changing the magnetic flux closed by the circuit. The discovery is made even more useful when extended to the general statement that a changing magnetic field produces an electric field. Such induced electric fields are very different from the fields produced by electric charges.

3.1 The Integral form of Faraday's Law

$$\begin{aligned} &\oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dl} \int_S \vec{B} \cdot \hat{n}\ da \\ &\oint_C \vec{E} \cdot \vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial l} \cdot \hat{n}\ da \quad \text{(Faraday's law alternative form)} \end{aligned} \tag{5}$$

Changing magnetic flux through a surface induces an emf(electromotive force) in any boundary path of the surface and changing magnetic flux field induces circulating electric field.

If magnetic flux through a surface changes, then an electric field (circulating electric field) induces along the boundary of that surface and if we place a conducting wire along the boundary, then current will flow.

- Induced Electric field Induced electric field is similar to electrostatic field but different in nature.
  case I: Electrostatic field Electrostatics field is charge base field for which positive charge is the originating point and negative charge is the terminating point. Electrostatic field is conservative in nature. for electrostatic field
  $$\vec{\nabla} \cdot \vec{E} \neq 0$$
  case II: Induced Electric field Induced electric field is produced by changing magnetic flux. Here, there is no originating and terminating point. The induced electric electric field forms loops and non conservative in nature.
  $$\vec{\nabla} \cdot \vec{E} = 0$$

$$\oint_C \vec{E} \cdot \vec{dl} \quad \rightarrow \quad \left(\begin{array}{l} \text{Circulating electric field is present} \\ \text{everywhere in the path } C \text{ and} \\ \text{capable of driving charge.} \end{array} \right).$$

Note: Magnetic flux
$$\phi_{\vec{B}} = \int_S \vec{B} \cdot \hat{n}\, da \tag{6}$$
If you think that this quantity in equation (6) must be zero according to Gauss's law for magnetic fields, then let me tell you, if you look this quantity more carefully, then the integral in this expression is over any surface, whereas the integral in Gauss's law is specifically over a closed surface. The magnetic flux through an open surface may indeed be non-zero. It is only when the surface is closed, the number of magnetic field lines passing through the surface in one direction must equal to the number passing through in the other direction.

- **Lenz's Law** The name comes from Heinrich Lenz, a German physicist who had an important insight concerning the direction of current induced by changing magnetic flux.
  current induced by a changing magnetic flux always flows in a direction that opposes the change in flux.

There are three ways to study the change in magnetic flux.

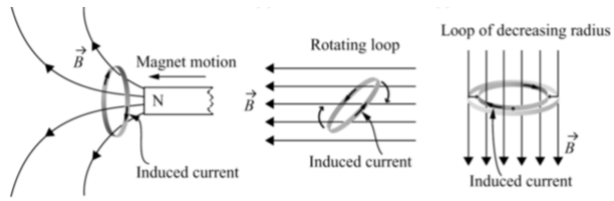

Figure 1: Changing Magnetic Flux and Induced Current

3.2 The differential form of Faraday's law

A circulating electric field is produced by a magnetic field that changes with time.

$$\vec{\nabla} \cdot \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

The left side of the equation is a mathematical description of curl of electric field, which is the tendency of the field lines to circulate around a point, and the right side represents the rate of change of magnetic field over time.

4 The Ampere-Maxwell Law

For thousands of years, the only known sources of magnetic fields were certain iron ores and other materials that had been magnetized. It was **Hans Christian Oersted** who deflected a compass needle by passing an electric current nearby and then, **Ampere** had begun quantifying the relationship between electric current and magnetic fields. Ampere's law relates a steady electric current to a circulating magnetic field and then **James Clerk Maxwell** began his work in the field in the 1850s. Ampere's law was only known to apply static situations involving steady current. It was Maxwell's addition of another source term, a changing electric flux that extended the applicability of Ampere's law to time-dependent conditions. Now this is called ***Ampere's-Maxwell law*** that allowed Maxwell to discern the electromagnetic nature of light and to develop the comprehensive theory of electromagnetism.

4.1 The Integral Form of Ampere-Maxwell Law

A changing electric flux through a surface produces a circulating magnetic field around any path that bounded that surface.

$$\oint_c \vec{B} \cdot \vec{dl} = \mu_0 \, I_{enc} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} \, da \tag{8}$$

We have the mathematical description of circulation of magnetic field around a closed path C in the left side of the equation (8) and the right side includes two sources for magnetic field, a **steady conduction current** and a **changing electric flux** through any surface as bounded by path C. Magnetic fields induced by changing electric flux are extremely weak.Therefore, it is very difficult to measure.

since no conduction current flows between the capacitor plates, what else might be going on in that region that would serve as the source of a magnetic field?

note: **Displacement Current and Changing Magnetic Flux**
The original form of Ampere's Law stated that the circulation of the magnetic field around a closed loop is proportional to the total current passing through it. However, a problem arose when applying this law to a charging capacitor. If we consider an Amperical loop around a wire leading to the capacitor, the law correctly predicts a magnetic field due to the conduction current. But if we take a surface passing through the gap between the capacitor plates, no conduction current is present, and according to the original Ampere's Law, there should be no magnetic field, which contradicts experimental observations. To resolve this, Maxwell introduced the concept of displacement current, recognizing that the changing electric field between the capacitor plates creates a changing electric flux, which effectively behaves like a current. He modified Ampere's Law by adding a term for this displacement current.

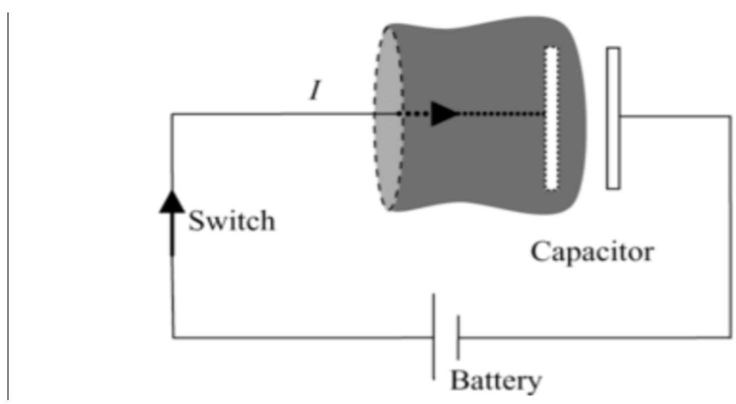

Figure 2: capacitor circuit

$$I_d = \epsilon_0 \frac{\partial \phi_{\vec{E}}}{\partial t}$$

Where I_d is the displacement current.
Then the final equation look like the following...

$$\oint_c \vec{B} \cdot \vec{dl} = \mu_0 \; I_{enc} + \mu_0 I_d$$

4.2 The Differential form of the Ampere–Maxwell law

A circulating magnetic field is produced by an electric current and an electric field that changes with time.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{9}$$

where,

$$I = \int_S \vec{J}\, \hat{n}\, da$$

The left side of the equation (9) is the curl of the magnetic field (the tendency of the field lines to circulate around a point).The two terms on the right side represent the electric current density (the amount of current through a cross section of the conductor) and the time rate of change of the electric field.