Magnetostatics

Shubham Yadav

Abstract

This lecture provides an in-depth discussion on magnetostatics, focusing on the magnetic fields generated by moving charges and currents. It begins with Coulomb's law and the distinction between electrostatic and magnetic forces, followed by the Lorentz force and the motion of charged particles in a magnetic field, including circular and helical trajectories. The concepts of current, surface, and volume current densities are introduced, leading to the Biot-Savart law and Ampère's law for calculating magnetic fields. The session concludes with an analysis of the cyclotron, explaining its working principle, frequency conditions, and limitations in accelerating charged particles.

BLS01(Bose Lecture Series), Theoretical Nexus

Introduction:

We have collection of charges q_1, q_2, q_3, ...(Source Charges) and a test charge Q. Coulomb's law gives the force on Q by individual charges and total force is the vector sum of all individual forces (i.e. *Principal of Superposition).*

The total force can be given by...

$$\vec{F}_{net} = \sum_{i=1}^{n} \frac{Q}{4\pi\epsilon_0} \cdot \frac{q_i}{\vec{r}_i^2} \hat{r}$$

• force between charges in motion

whatever force accounts for the attraction of parallel currents and the repulsion of antiparallel current is not electrostatic in nature. This is some other force which exist due to the motion of the charges.This is our first encounter with magnetic force.

Note: Stationary charges produces only electric field $\vec{\mathbf{E}}$ *and a moving charge generates, in addition, a magnetic field* $\vec{\mathbf{B}}$ *in the space around it.*

1 Magnetic field(B):

The region around a magnetic material or magnetic charge within which force of magnetism act.

Direction: *to detect the direction of the* $\vec{\mathbf{B}}$ *all you need is Boy Scout compass. Here it is important to know that that compass points in the direction of local magnetic field because in lab typical field may be hundreds of time stronger than earth's magnetic field.*

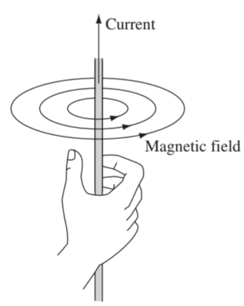

Figure 1: Magnetic Field Lines

1.1 Magnetic Force:

The magnetic force on the charge particle Q, moving with velocity $\vec{v}$ *in a magnetic field* $\vec{B}$*.*

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$$

in magnetic field charge Q will experience $\vec{v} \times \vec{B}$ *force on it and the direction can be given by cross product(*$\vec{v} \times \vec{B}$*). This is known as* **Lorentz force law**.

In the presence of both electric field and magnetic field the net force on charge Q will be

$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})$$

1.2 Motion of Charge Particle in $\vec{B}$:

Magnetic force $(\vec{v} \times \vec{B})$ *provides centripetal acceleration to the charge particle when it will enters perpendicular magnetic field.*

$$\vec{F}_{mag} = \vec{F}_{centripetel}$$

$$Q(\vec{v} \times \vec{B}) = \frac{m\vec{v}^2}{R}$$

$$Q(vB \cdot sin\theta) = \frac{mv^2}{R} \quad \left(where\ \theta = \frac{\pi}{2}\right)$$

$$\frac{v}{R} = \frac{QB}{m}$$

Radius of the charge particle in which it move

$$R = \frac{mv}{QB} = \frac{\sqrt{2mK}}{QB}$$

Where K is the Kinetic Energy of the charge particle

Time period $T = \frac{2\pi m}{QB}$

1.3 Helical Path

In the given situation the component of velocity($v\,cos\theta$*) which is parallel to the magnetic field will not be affected and only the vertical component will be affected by magnetic force and causes the circular path for the particle to move and the radius of that circular path will be* $\frac{mvsin\theta}{QB}$

`trajectory of particle:` HELICAL = STRAIGHT + CIRCULAR ***Pitch of Helical Path:*** *Horizontal distance covered in one time period.*

$$pitch = v_{||} \cdot (time\ in\ one\ cycle)$$

$$pitch = vcos\theta \cdot \frac{2\pi m}{QB}$$

2 Current:

Current in a wire is the charge per unit time passing to a given point.

$$I = \frac{Q}{\Delta t}$$

By convention, negative charges moving determine the same current as positive charges do in opposite direction. A linear charge density λ travelling in a wire at speed v constitutes a current I

$$I = \frac{Q}{\Delta t}$$

$$I = \frac{\lambda dl}{\Delta t} = \lambda \frac{dl}{\Delta t} = \lambda v$$

$$I = \lambda v$$

since,

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) dq$$

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) \lambda dl$$

$$\vec{F}_{mag} = \int (\lambda \vec{v} \times \vec{B}) dl$$

$$\vec{F}_{mag} = I \int (\vec{dl} \times \vec{B})$$

Current Density

1. Surface Current Density

When charge flows over a surface, we describe it by surface current density,K. That means K is the current per unit width.

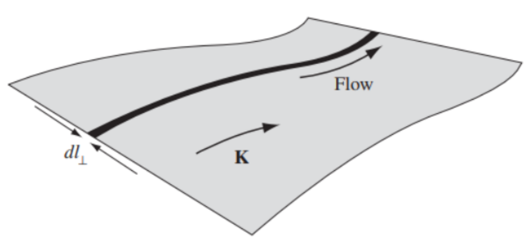

Figure 2: Surface Current Density

$$k = \frac{I}{dL_{\perp}}$$

K in terms of σ and v

$$\sigma = \frac{dQ}{dA} = \frac{dQ}{dx \cdot dl}$$

$$dI = \frac{dQ}{dt}$$

$$\sigma = \frac{dQ}{dx \cdot dL}$$

$$\sigma \, dx \, dL = dQ$$

$$I = \frac{dQ}{dt} = \frac{\sigma \, dx \, dl_{\perp}}{dt}$$

$$K = \frac{\sigma dx \, dl_{\perp}}{dt \, dl_{\perp}} = \sigma \vec{v}$$

$$K = \sigma \vec{v}$$

$$\vec{F} = \int (\vec{v} \times \vec{B}) \sigma dA$$

$$\vec{F} = \int (\sigma \vec{v} \times \vec{B}) dA$$

$$\vec{F} = \int (\vec{K} \times \vec{B}) dA$$

2. Volume Current Density

When the flow of charge is distributed throughout a three-dimensional region, then suck distribution is called the volume current density,J.

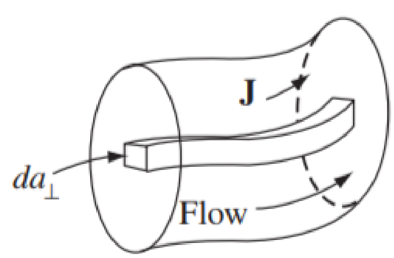

Figure 3: Volume Current Density

$$J = \frac{I}{da_\perp}$$

similarly,

$$J = \rho \vec{v}$$

and

$$F = \int (\vec{v} \times \vec{B}) \sigma d\tau$$

$$F = \int (\vec{J} \times \vec{B}) d\tau$$

3 The Biot-Savart Law

3.1 Steady Current

Stationary charges produce electric field that is constant in time. Similarly, steady current produce magnetic field that is constant in time. Now, what is the steady current? steady current is the continuous flow of charge that has been going on forever, without change and without piling up anywhere. that is,

$$\frac{\delta \rho}{\delta t} = 0, \frac{\delta J}{\delta t} = 0$$

in reality there is no such thing as a true steady current but there exist true positive charges.

The Magnetic Field of a Steady Current

The magnetic field due to a steady line current is given by the ***Biot-Savart Law****:*

$$d\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{\vec{dl} \times \hat{r}}{r^2} \tag{1}$$

where;

The constant μ_0 is the permeability of free space, given by

$\mu_0 = 4\pi \times 10^{-7}\ N/A^2$. *$I$ is the steady current flowing through the wire. $d\mathbf{l}$ is a small element of the current path. $\hat{\mathbf{r}}$ is the unit vector from the current element to the point of observation. r is the distance between the current element and the observation point.*

The direction of the integration follows the current path, with $d\mathbf{l}$ oriented along the flow of the current.

The SI unit of the magnetic field is the ***Tesla (T)****, where:*

$$1T = 1 \frac{N}{A \cdot m} \tag{2}$$

This law provides a fundamental way to calculate the magnetic field produced by any current distribution by integrating over the entire current path.

Ampère's Law

The line integral of $\vec{B}$ around a closed path is called Ampère's law.

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

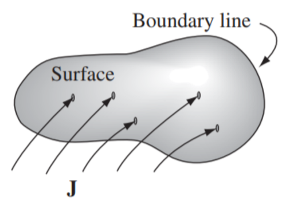

Figure 4: Ampère's Circuital Law

enclosed current is all currents that penetrate any surface for which path C is a boundary.

Cyclotron: Construction, Working, and Uses

Definition: *A cyclotron is a device used to accelerate charged particles to very high speeds (and hence high energies). It is commonly used:*

(i) *To study nuclear reactions.*

(ii) *To study radioactive decay.*

(iii) *TO insert an ion in a solid.*

Construction

- *Two hollow, semi-circular metal containers called* Dees *(because of their 'D' shape).*
- *These Dees are placed back-to-back with a small gap between them.*
- *A high-frequency oscillator is connected across the Dees, providing an alternating electric field i youn the gap.*
- *The entire setup is placed in a uniform magnetic field* **B** *, oriented perpendicular to the plane of the Dees.*
- *A source injects charged particles (e.g. protons) near the center of the Dees.*

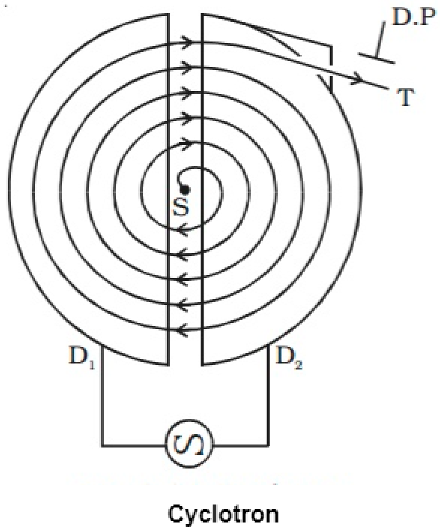

Figure 5: Cyclotron

Working Principle

A charged particle is introduced at the center (the gap between the Dees). The alternating electric field in the gap accelerates the particle each time it crosses from one Dee to the other. Inside each Dee, the magnetic field forces the particle to move in a semicircular path (no electric field inside the metal Dee), so the speed remains constant within *a Dee. Every time the particle reaches the gap, the electric field* reverses *just in time to accelerate the particle again, increasing its speed. As speed increases, the radius of the circular path inside each Dee grows. Eventually, at a large enough radius, the particle is extracted from cyclotron at very high speed.*

Time Period and Frequency of Q and AC

Time period and the frequency of the charge particle doesn't depend upon velocity and radius of charge, therefore each spiral path have the same time period and the frequency.

$$T_q = \frac{2\pi m}{Q\vec{B}}, \qquad f_q = \frac{Q\vec{B}}{2\pi m}$$

therefore, we have to set the frequency of AC oscillator exactly equal to the frequency of charge.

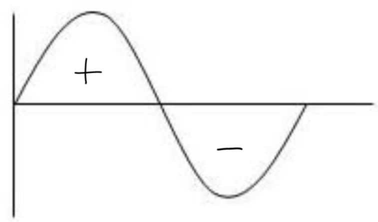

Figure 6: One Complete Cycle of AC

In one complete circle of the charge, oscillator should give one positive and one negative cycle that has to be equal to the one cycle of AC.

$$frequency\ of\ charge = frequency\ of\ AC$$

$$f_q = f_{AC} = \frac{Q\vec{B}}{2\pi m}$$

Limitations of the Cyclotron

1. ***Neutral particles cannot be accelerated***
   Since cyclotrons rely on electromagnetic forces, they cannot accelerate uncharged particles such as neutrons.

2. ***very small charge particles***
   it can not accelerate very small charge particles like electron because in that case massm will vary. since,

   $$\vec{F} = Q\,\vec{E}$$

   $$\vec{a} = \frac{q\,\vec{E}}{m} \qquad (m\uparrow, \vec{a}\downarrow, \vec{v}\uparrow)$$

   $$m = \frac{m_0}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}$$

   Since mass of the particle is very small than velocity will be very large and when velocity will reach the the order of speed of light then mass will not be the constant but increase and time period of charge particle will also vary.

   $$\mathbf{T}(\uparrow) = \frac{2\pi\mathbf{m}(\uparrow)}{q\vec{B}}$$

3. *It can not accelerate particles up to very high speed as compared to the speed of light.*